Solving Nonlinear Wave Equations and Lattices with Mathematica

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Purpose & Motivation

• Develop various symbolic algorithms to compute exact solutions of nonlinear (systems) of partial differential equations (PDEs) and differential-difference equations (DDEs, lattices).

• Fully automate the tanh and sech methods to compute closed form solitary wave solutions.

• Solutions of tanh or sech type model solitary waves in fluid dynamics, plasmas, electrical circuits, optical fibers, bio-genetics, etc.

• Class of nonlinear PDEs and DDEs solvable with the tanh/sech method includes famous evolution and wave equations. Typical examples: Korteweg-de Vries, Fisher and Boussinesq PDEs, Toda and Volterra lattices (DDEs).

• Research aspect: Design a high-quality application package for the computation of exact solitary wave solutions of large classes of nonlinear evolution and wave equations.

• Educational aspect: Software as course ware for courses in nonlinear PDEs, theory of nonlinear waves, integrability, dynamical systems, and modeling with symbolic software.

• Users: scientists working on nonlinear wave phenomena in fluid dynamics, nonlinear networks, elastic media, chemical kinetics, material science, bio-sciences, plasma physics, and nonlinear optics.
Typical Examples of Single PDEs and Systems of PDEs

- The Korteweg-de Vries (KdV) equation:
  \[ u_t + \alpha uu_x + u_{3x} = 0. \]
  Solitary wave solution:
  \[ u(x, t) = \frac{8c_1^3 - c_2}{6\alpha c_1} - \frac{2c_1^2}{\alpha} \tanh^2 [c_1 x + c_2 t + \delta], \]
  or, equivalently,
  \[ u(x, t) = -\frac{4c_1^3 + c_2}{6\alpha c_1} + \frac{2c_1^2}{\alpha} \sech^2 [c_1 x + c_2 t + \delta]. \]

- The modified Korteweg-de Vries (mKdV) equation:
  \[ u_t + \alpha u^2 u_x + u_{3x} = 0. \]
  Solitary wave solution:
  \[ u(x, t) = \pm \sqrt{\frac{6}{\alpha} c_1} \sech [c_1 x - c_1^3 t + \delta]. \]

- The Fisher equation:
  \[ u_t - u_{xx} - u(1 - u) = 0. \]
  Solitary wave solution:
  \[ u(x, t) = \frac{1}{4} \pm \frac{1}{2} \tanh \xi + \frac{1}{4} \tanh^2 \xi, \]
  with
  \[ \xi = \pm \frac{1}{2\sqrt{6}} x \pm \frac{5}{12} t + \delta. \]
• The generalized Kuramoto-Sivashinski equation:

\[ u_t + uu_x + u_{xx} + \sigma u_{3x} + u_{4x} = 0. \]

Solitary wave solutions
(ignoring symmetry \( u \to -u, x \to -x, \sigma \to -\sigma \)):
For \( \sigma = 4 \):

\[ u(x, t) = 9 - 2c^2 - 15 \tanh\xi (1 + \tanh\xi - \tanh^2\xi), \]
with \( \xi = \frac{x}{2} + c^2 t + \delta. \)

For \( \sigma = 0 \):

\[ u(x, t) = -2 \sqrt{\frac{19}{11}} c^2 - \frac{135}{19} \sqrt{\frac{11}{19}} \tanh\xi + \frac{165}{19} \sqrt{\frac{11}{19}} \tanh^3\xi, \]
with \( \xi = \frac{1}{2} \sqrt{\frac{11}{19}} x + c^2 t + \delta. \)

For \( \sigma = \frac{12}{\sqrt{47}} \):

\[
\begin{align*}
  u(x, t) &= \frac{45 \mp 4418c^2}{47\sqrt{47}} \pm \frac{45}{47\sqrt{47}} \tanh\xi \\
  &\quad - \frac{45}{47\sqrt{47}} \tanh^2\xi \pm \frac{15}{47\sqrt{47}} \tanh^3\xi,
\end{align*}
\]
with \( \xi = \pm \frac{1}{2\sqrt{47}} x + c^2 t + \delta. \)

For \( \sigma = 16/\sqrt{73} \):

\[
\begin{align*}
  u(x, t) &= \frac{2 (30 \mp 5329c^2)}{73\sqrt{73}} \pm \frac{75}{73\sqrt{73}} \tanh\xi \\
  &\quad - \frac{60}{73\sqrt{73}} \tanh^2\xi \pm \frac{15}{73\sqrt{73}} \tanh^3\xi,
\end{align*}
\]
with \( \xi = \pm \frac{1}{2\sqrt{73}} x + c^2 t + \delta. \)
• Three-dimensional modified Korteweg-de Vries equation:

\[ u_t + 6u^2u_x + u_{xyz} = 0. \]

Solitary wave solution:

\[ u(x, y, z, t) = \pm \sqrt{c_2 c_3} \text{sech} [c_1 x + c_2 y + c_3 z - c_1 c_2 c_3 t + \delta]. \]

• The Boussinesq (wave) equation:

\[ u_{tt} - \beta u_{2x} + 3uu_{2x} + 3u_x^2 + \alpha u_{4x} = 0, \]

or written as a first-order system \((v)\) auxiliary variable):

\[ u_t + v_x = 0, \]
\[ v_t + \beta u_x - 3uu_x - \alpha u_{3x} = 0. \]

Solitary wave solution:

\[ u(x, t) = \frac{\beta c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2 [c_1 x + c_2 t + \delta] \]
\[ v(x, t) = b_0 + 4\alpha c_1 c_2 \tanh^2 [c_1 x + c_2 t + \delta]. \]

• The Broer-Kaup system:

\[ u_{ty} + 2(uu_x)_y + 2v_{xx} - u_{xxy} = 0, \]
\[ v_t + 2(uv)_x + v_{xx} = 0. \]

Solitary wave solution:

\[ u(x, t) = -\frac{c_3}{2c_1} + c_1 \tanh [c_1 x + c_2 y + c_3 t + \delta] \]
\[ v(x, t) = c_1 c_2 - c_1 c_2 \tanh^2 [c_1 x + c_2 y + c_3 t + \delta] \]
Typical Examples of DDEs (lattices)

- The Toda lattice:
  \[ \ddot{u}_n = (1 + \dot{u}_n)(u_{n-1} - 2u_n + u_{n+1}) \,.
\]
  Solitary wave solution:
  \[ u_n(t) = a_0 \pm \sinh(c_1) \tanh [c_1 n \pm \sinh(c_1) t + \delta] \,.
\]

- The Volterra lattice:
  \[ \dot{u}_n = u_n(v_n - v_{n-1}) \]
  \[ \dot{v}_n = v_n(u_{n+1} - u_n) \,.
\]
  Solitary wave solution:
  \[ u_n(t) = -c_2 \coth(c_1) + c_2 \tanh [c_1 n + c_2 t + \delta] \]
  \[ v_n(t) = -c_2 \coth(c_1) - c_2 \tanh [c_1 n + c_2 t + \delta] \,.
\]

- The Relativistic Toda lattice:
  \[ \dot{u}_n = (1 + \alpha u_n)(v_n - v_{n-1}) \]
  \[ \dot{v}_n = v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}) \,.
\]
  Solitary wave solution:
  \[ u_n(t) = -c_2 \coth(c_1) - \frac{1}{\alpha} + c_2 \tanh [c_1 n + c_2 t + \delta] \]
  \[ v_n(t) = \frac{c_2 \coth(c_1)}{\alpha} - \frac{c_2}{\alpha} \tanh [c_1 n + c_2 t + \delta] \,.
\]
Algorithm for Tanh Solutions for PDE system

Given is a system of PDEs of order n

$$\Delta(u(x), u'(x), u''(x), \cdots u^{(n)}(x)) = 0.$$  

Dependent variable $u$ has components $u_i$ (or $u, v, w, \ldots$)

Independent variable $x$ has components $x_i$ (or $x, y, z, t$)

**Step 1:**

- Seek solution of the form $u_i(x) = U_i(T)$, with
  $$T = \tanh[c_1x + c_2y + c_3z + c_4t] = \tanh \xi.$$  

- Observe $\cosh^2 \xi - \sinh^2 \xi = 1$, $(\tanh \xi)' = 1 - \tanh^2 \xi$
  or $T' = 1 - T^2$.

- Repeatedly apply the operator rule
  $$\frac{\partial \bullet}{\partial x_i} \rightarrow c_i(1 - T^2) \frac{d \bullet}{dT}$$

This produces a coupled system of Legendre equations of type

$$P(T, U_i, U_i', \ldots, U_i^{(n)}) = 0$$

for $U_i(T)$.

- Example: For Boussinesq system
  $$u_t + v_x = 0$$
  $$v_t + \beta u_x - 3uu_x - \alpha u_{3x} = 0,$$

we obtain after cancelling common factors $1 - T^2$

$$c_2U' + c_1V' = 0$$

$$c_2V' + \beta c_1U' - 3c_1UU'$$
$$+ \alpha c_1^3 \left[2(1 - 3T^2)U' + 6T(1 - T^2)U'' - (1 - T^2)^2U''' \right] = 0$$


Step 2:

- Seek polynomial solutions

\[ U_i(T) = \sum_{j=0}^{M_i} a_{ij} T^j \]

Balance the highest power terms in \( T \) to determine \( M_i \).

- Example: Powers for Boussinesq system

\[ M_1 - 1 = M_2 - 1, \quad 2M_1 - 1 = M_1 + 1 \]

gives \( M_1 = M_2 = 2 \).

Hence, \( U_1(T) = a_{10} + a_{11} T + a_{12} T^2 \), \( U_2(T) = a_{20} + a_{21} T + a_{22} T^2 \).

Step 3:

- Determine the algebraic system for the unknown coefficients \( a_{ij} \) by balancing the coefficients of the various powers of \( T \).

- Example: Boussinesq system

\[
\begin{align*}
 a_{11} c_1 (3a_{12} + 2\alpha c_1^2) &= 0 \\
a_{12} c_1 (a_{12} + 4\alpha c_1^2) &= 0 \\
a_{21} c_1 + a_{11} c_2 &= 0 \\
a_{22} c_1 + a_{12} c_2 &= 0 \\
\beta a_{11} c_1 - 3a_{10} a_{11} c_1 + 2\alpha a_{11} c_1^3 + a_{21} c_2 &= 0 \\
-3a_{11} c_1 + 2\beta a_{12} c_1 - 6a_{10} a_{12} c_1 + 16\alpha a_{12} c_1^3 + 2a_{22} c_2 &= 0.
\end{align*}
\]
Step 4:

• Solve the nonlinear algebraic system with parameters.
  Reject complex solutions? Test the solutions.

• Example: Solution for Boussinesq case

\[
\begin{align*}
a_{10} &= \frac{\beta c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} \\
a_{11} &= 0 \\
a_{12} &= -4\alpha c_1^2 \\
a_{20} &= \text{free} \\
a_{21} &= 0 \\
a_{22} &= 4\alpha c_1 c_2.
\end{align*}
\]

Step 5:

• Return to the original variables.
  Test the final solution in the original equations

• Example: Solitary wave solution for Boussinesq system:

\[
\begin{align*}
u(x, t) &= \frac{\beta c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2 [c_1 x + c_2 t + \delta] \\
v(x, t) &= a_{20} + 4\alpha c_1 c_2 \tanh^2 [c_1 x + c_2 t + \delta].
\end{align*}
\]
Algorithm for Sech Solutions for PDE system

Given is a system of PDEs of order n
\[ \Delta(u(x), u'(x), u''(x), \cdots u^{(n)}(x)) = 0. \]

Dependent variable \( u \) has components \( u_i \) (or \( u, v, w, \ldots \))
Independent variable \( x \) has components \( x_i \) (or \( x, y, z, t \))

**Step 1:**
- Seek solution of the form \( u_i(x) = U_i(S) \), with
  \[ S = \text{sech} [c_1 x + c_2 y + c_3 z + c_4 t] = \tanh \xi. \]
- Observe \( (\text{sech} \xi)' = -\tanh \xi \text{sech} \xi \) or \( S' = -TS = -\sqrt{1 - S^2} \).
  Also, \( \cosh^2 \xi - \sinh^2 \xi = 1 \), hence, \( T^2 + S^2 = 1 \) and \( \frac{dT}{dS} = -\frac{S}{T} \).
- Repeatedly apply the operator rule
  \[ \frac{\partial \bullet}{\partial x_i} \rightarrow -c_i \sqrt{1 - S^2} S \frac{d \bullet}{dS} \]
  This produces a coupled system of Legendre type equations of type
  \[ P(S, U_i, U'_i, \ldots, U^{(m)}_i) + \sqrt{1 - S^2} Q(S, U_i, U'_i, \ldots, U^{(n)}_i) = 0 \]
  for \( U_i(S) \).
  For every equation one must have \( P_i = 0 \) or \( Q_i = 0 \). Only odd derivatives produce the extra factor \( \sqrt{1 - S^2} \).
  Conclusion: The total number of derivatives in each term in the given system should be either even or odd. No mismatch is allowed.
- Example: For the 3D mKdV equation
  \[ u_t + 6u^2u_x + u_{xyz} = 0. \]
we obtain after cancelling a common factor $-\sqrt{1-S^2} S$
\[ c_4 U' + 6c_1 U^2 U' + c_1 c_2 c_3 [(1-6S^2)U' + 3S(1-2S^2)U'' + S^2(1-S^2)U'''] = 0 \]

**Step 2:**

- Seek polynomial solutions
  \[ U_i(S) = \sum_{j=0}^{M_i} a_{ij} S^j \]
  Balance the highest power terms in $S$ to determine $M_i$.

- Example: Powers for the 3D mKdV case
  \[ 3M_1 - 1 = M_1 + 1 \]
  gives $M_1 = 1$. Hence, $U(S) = a_{10} + a_{11} S$.

**Step 3:**

- Determine the algebraic system for the unknown coefficients $a_{ij}$ by balancing the coefficients of the various powers of $S$.

- Example: System for 3D mKdV case
  \begin{align*}
  a_{11} c_1 (a_{11}^2 - c_2 c_3) &= 0 \\
  a_{11} (6a_{10}^2 c_1 + c_1 c_2 c_3 + c_4) &= 0 \\
  a_{10} a_{11}^2 c_1 &= 0
  \end{align*}
Step 4:

- Solve the nonlinear algebraic system with parameters.
  Reject complex solutions? Test the solutions.
- Example: Solution for 3D mKdV case
  \[ a_{10} = 0 \]
  \[ a_{11} = \pm \sqrt{c_1 c_3} \]
  \[ c_4 = -c_1 c_2 c_3 \]

Step 5:

- Return to the original variables.
  Test the final solution in the original equations
- Example: Solitary wave solution for the 3D mKdV equation
  \[ u(x, y, z, t) = \pm \sqrt{c_2 c_3} \text{sech}(c_1 x + c_2 y + c_3 z - c_1 c_2 c_3 t). \]
Extension: Tanh Solutions for DDE system

Given is a system of differential-difference equations (DDEs) of order \( n \)
\[
\Delta(..., u_{n-1}, u_n, u_{n+1}, ..., \dot{u}_n, ..., u^{(m)}_n, ...) = 0.
\]

Dependent variable \( u_n \) has components \( u_{i,n} \) (or \( u_n, v_n, w_n, ... \))
Independent variable \( x \) has components \( x_i \) (or \( n, t \)).
No derivatives on shifted variables are allowed!

**Step 1:**
- Seek solution of the form \( u_{i,n}(x) = U_{i,n}(T(n)) \), with
  \[
  T(n) = \tanh [c_1 n + c_2 t + \delta] = \tanh \xi.
  \]
- Note that the argument \( T \) depends on \( n \). Complicates matters.
- Repeatedly apply the operator rule on \( u_{i,n} \)
  \[
  \frac{\partial \bullet}{\partial t} \rightarrow c_2(1 - T^2) \frac{d \bullet}{dT}
  \]
  This produces a coupled system of Legendre equations of type
  \[
  P(T, U_{i,n}, U'_{i,n}, ...) = 0
  \]
  for \( U_{i,n}(T) \).
- Example: Toda lattice
  \[
  \ddot{u}_n = (1 + \dot{u}_n)(u_{n-1} - 2u_n + u_{n+1}).
  \]
  transforms into
  \[
  c_2^2(1 - T^2) \left[ 2TU'_n - (1 - T^2)U''_n \right] \\
  + \left[ 1 + c_2(1 - T^2)U'_n \right] [U_{n-1} - 2U_n + U_{n+1}] = 0
  \]
Step 2:

- Seek polynomial solutions

\[ U_{i,n}(T(n)) = \sum_{j=0}^{M_i} a_{ij} T(n)^j \]

For \( U_{n+p} \), \( p \neq 0 \), there is a phase shift:

\[ U_{i,n\pm p}(T(n \pm p)) = \sum_{j=0}^{M_i} a_{i,j} [T(n + p)]^j = \sum_{j=0}^{M_i} a_{i,j} \left[ \frac{T(n) \pm \tanh(pc_1)}{1 \pm T(n) \tanh(pc_1)} \right]^j \]

Balance the highest power terms in \( T(n) \) to determine \( M_i \).

- Example: Powers for Toda lattice

\[ 2M_1 - 1 = M_1 + 1 \]

gives \( M_1 = 1 \).

Hence,

\[ U_n(T(n)) = a_{10} + a_{11} T(n) \]
\[ U_{n-1}(T(n - 1)) = a_{10} + a_{11} T(n - 1) = a_{10} + a_{11} \frac{T(n) - \tanh(c_1)}{1 - T(n) \tanh(c_1)} \]
\[ U_{n+1}(T(n + 1)) = a_{10} + a_{11} T(n + 1) = a_{10} + a_{11} \frac{T(n) + \tanh(c_1)}{1 + T(n) \tanh(c_1)} \].

Step 3:

- Determine the algebraic system for the unknown coefficients \( a_{ij} \) by balancing the coefficients of the various powers of \( T(n) \).

- Example: Algebraic system for Toda lattice

\[ c_2^2 - \tanh^2(c_1) - a_{11}c_2 \tanh^2(c_1) = 0, \quad c_2 - a_{11} = 0 \]
Step 4:

- Solve the nonlinear algebraic system with parameters.
  Reject complex solutions? Test the solutions.
- Example: Solution of algebraic system for Toda lattice
  \[ a_{10} = \text{free}, \quad a_{11} = c_2 = \pm \sinh(c_1) \]

Step 5:

- Return to the original variables.
  Test the final solution in the original equations
- Example: Solitary wave solution for Toda lattice:
  \[ u_n(t) = a_0 \pm \sinh(c_1) \tanh[c_1 n \pm \sinh(c_1) t + \delta]. \]
Example: System of DDEs: Relativistic Toda lattice

\[
\begin{align*}
\dot{u}_n & = (1 + \alpha u_n)(v_n - v_{n-1}) \\
\dot{v}_n & = v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}).
\end{align*}
\]

**Step 1**: Change of variables

\[
U_n(x, t) = U_n(T(n)), \quad V_n(x, t) = V_n(T(n)),
\]

with

\[
T(n) = \tanh [c_1 n + c_2 t + \delta] = \tanh \xi.
\]

gives

\[
\begin{align*}
c_2(1 - T^2)U'_n - (1 + \alpha U_n)(V_n - V_{n-1}) & = 0 \\
c_2(1 - T^2)V'_n - V_n(U_{n+1} - U_n + \alpha V_{n+1} - \alpha V_{n-1}) & = 0.
\end{align*}
\]

**Step 2**: Seek polynomial solutions

\[
\begin{align*}
U_n(T(n)) & = \sum_{j=0}^{M_1} a_{1j} T(n)^j \\
V_n(T(n)) & = \sum_{j=0}^{M_2} a_{2j} T(n)^j.
\end{align*}
\]

Balance the highest power terms in \(T(n)\) to determine \(M_1\), and \(M_2\):

\[
M_1 + 1 = M_1 + M_2, \quad M_2 + 1 = M_1 + M_2
\]

gives \(M_1 = M_2 = 1\).

Hence,

\[
U_n = a_{10} + a_{11} T(n), \quad V_n = a_{20} + a_{21} T(n).
\]
Step 3: Algebraic system for $a_{ij}$:

\[-a_{11} c_2 + a_{21} \tanh(c_1) + \alpha a_{10} a_{21} \tanh(c_1) = 0\]
\[a_{11} \tanh(c_1) (\alpha a_{21} + c_2) = 0\]
\[-a_{21} c_2 + a_{11} a_{20} \tanh(c_1) + 2\alpha a_{20} a_{21} \tanh(c_1) = 0\]
\[\tanh(c_1) (a_{11} a_{21} + 2\alpha a_{21}^2 - a_{11} a_{20} \tanh(c_1)) = 0\]
\[a_{21} \tanh^2(c_1) (c_2 - a_{11}) = 0\]

Step 4: Solution of the algebraic system

\[a_{10} = -c_2 \coth(c_1) - \frac{1}{\alpha}, \quad a_{11} = c_2, \quad a_{20} = \frac{c_2 \coth(c_1)}{\alpha}, \quad a_{21} = -\frac{c_2}{\alpha}.\]

Step 5: Solitary wave solution in original variables:

\[u_n(t) = -c_2 \coth(c_1) - \frac{1}{\alpha} + c_2 \tanh[c_1 n + c_2 t + \delta]\]
\[v_n(t) = \frac{c_2 \coth(c_1)}{\alpha} - \frac{c_2}{\alpha} \tanh[c_1 n + c_2 t + \delta].\]
Solving/Analyzing Systems of Algebraic Equations with Parameters

Class of fifth-order evolution equations with parameters:

\[ u_t + \alpha \gamma^2 u^2 u_x + \beta \gamma u_x u_{2x} + \gamma uu_{3x} + u_{5x} = 0. \]

**Well-Known Special cases**

Lax case: \( \alpha = \frac{3}{10}, \beta = 2, \gamma = 10 \). **Two** solutions:

\[ u(x, t) = 4c_1^2 - 6c_1^2 \tanh^2 \left[ c_1 x - 56c_1^5 t + \delta \right] \]

and

\[ u(x, t) = a_0 - 2c_1^2 \tanh^2 \left[ c_1 x - 2(15a_0^2 c_1 - 40a_0 c_1^3 + 28c_1^5) t + \delta \right] \]

where \( a_0 \) is arbitrary.

Sawada-Kotera case: \( \alpha = \frac{1}{5}, \beta = 1, \gamma = 5 \). **Two** solutions:

\[ u(x, t) = 8c_1^2 - 12c_1^2 \tanh^2 \left[ c_1 x - 16c_1^5 t + \delta \right] \]

and

\[ u(x, t) = a_0 - 6c_1^2 \tanh^2 \left[ c_1 x - (5a_0^2 c_1 - 40a_0 c_1^3 + 76c_1^5) t + \delta \right] \]

where \( a_0 \) is arbitrary.

Kaup-Kupershmidt case: \( \alpha = \frac{1}{5}, \beta = \frac{5}{2}, \gamma = 10 \). **Two** solutions:

\[ u(x, t) = c_1^2 - \frac{3}{2}c_1^2 \tanh^2 \left[ c_1 x - c_1^5 t + \delta \right] \]

and

\[ u(x, t) = 8c_1^2 - 12c_1^2 \tanh^2 \left[ c_1 x - 176c_1^5 t + \delta \right] \]

no free constants!

Ito case: \( \alpha = \frac{2}{5}, \beta = 2, \gamma = 3 \). **One** solution:

\[ u(x, t) = 20c_1^2 - 30c_1^2 \tanh^2 \left[ c_1 x - 96c_1^5 t + \delta \right]. \]
What about the General case?

Q1: Can we retrieve the special solutions?

Q2: What are the condition(s) on the parameters $\alpha, \beta, \gamma$ for solutions of tanh-type to exist?

Tanh solutions:

$$u(x, t) = a_0 + a_1 \tanh [c_1 x + c_2 t + \delta] + a_2 \tanh^2 [c_1 x + c_2 t + \delta].$$

Nonlinear algebraic system must be analyzed, solved (or reduced!):

$$a_1(\alpha \gamma^2 a_2^2 + 6 \gamma a_2 c_1^2 + 2 \beta \gamma a_2 c_1^2 + 24 c_1^4) = 0$$

$$a_1(\alpha \gamma^2 a_1^2 + 6 \alpha \gamma^2 a_0 a_2 + 6 \gamma a_0 c_1^2 - 18 \gamma a_2 c_1^2 - 12 \beta \gamma a_2 c_1^2 - 120 c_1^4) = 0$$

$$\alpha \gamma^2 a_2^2 + 12 \gamma a_2 c_1^2 + 6 \beta \gamma a_2 c_1^2 + 360 c_1^4 = 0$$

$$2 \alpha \gamma^2 a_1^2 a_2 + 2 \alpha \gamma^2 a_0 a_2^2 + 3 \gamma a_1^2 c_1^2 + \beta \gamma a_1^2 c_1^2 + 12 \gamma a_0 a_2 c_1^2$$
$$-8 \gamma a_2^2 c_1^2 - 8 \beta \gamma a_2^2 c_1^2 - 480 a_2 c_1^4 = 0$$

$$a_1(\alpha \gamma^2 a_0^2 c_1 - 2 \gamma a_0 c_1^3 + 2 \beta \gamma a_2 c_1^3 + 16 c_1^5 + c_2) = 0$$

$$\alpha \gamma^2 a_0 a_1^2 c_1 + \alpha \gamma^2 a_0^2 a_2 c_1 - \gamma a_1^2 c_1^3 - \beta \gamma a_1^2 c_1^3 - 8 \gamma a_0 a_2 c_1^3 + 2 \beta \gamma a_2 c_1^3$$
$$+136 a_2 c_1^5 + a_2 c_2 = 0$$

Unknowns: $a_0, a_1, a_2$.

Parameters: $c_1, c_2, \alpha, \beta, \gamma$.

*Solve* and *Reduce* cannot be used on the whole system!
Strategy to Solve/Reduce Nonlinear Systems

Assumptions:

- All \( c_i \neq 0 \)
- Parameters \( (\alpha, \beta, \gamma, \ldots) \) are nonzero. Otherwise the highest powers \( M_i \) may change.
- All \( a_{jM_i} \neq 0 \). Coefficients of highest power in \( U_i \) are present.
- Solve for \( a_{ij} \), then \( c_i \), then find conditions on parameters.

Strategy followed by hand:

- Solve all linear equations in \( a_{ij} \) first (cost: branching). Start with the ones without parameters. Capture constraints in the process.
- Solve linear equations in \( c_i \) if they are free of \( a_{ij} \).
- Solve linear equations in parameters if they free of \( a_{ij}, c_i \).
- Solve quasi-linear equations for \( a_{ij}, c_i, \) parameters.
- Solve quadratic equations for \( a_{ij}, c_i, \) parameters.
- Eliminate cubic terms for \( a_{ij}, c_i, \) parameters, without solving.
- Show remaining equations, if any.

Alternatives:

- Use (adapted) Gröbner Basis Techniques.
- Use combinatorics on coefficients \( a_{ij} = 0 \) or \( a_{ij} \neq 0 \).
**Actual Solution**: Two major cases:

CASE 1: \( a_1 = 0 \), two subcases

**Subcase 1-a:**

\[
a_2 = \frac{3}{2}a_0
\]

\[
c_2 = c_1^3(24c_1^2 - \beta \gamma a_0)
\]

where \( a_0 \) is one of the two roots of the quadratic equation:

\[
\alpha \gamma^2 a_0^2 - 8\gamma a_0 c_1^2 - 4\beta \gamma a_0 c_1^2 + 160 c_1^4 = 0.
\]

**Subcase 1-b**: If \( \beta = 10\alpha - 1 \), then

\[
a_2 = -\frac{6}{\alpha \gamma} c_1^2
\]

\[
c_2 = -\frac{1}{\alpha}(\alpha^2 \gamma^2 a_0^2 c_1 - 8\alpha \gamma a_0 c_1^3 + 12 c_1^5 + 16\alpha c_1^5)
\]

where \( a_0 \) is arbitrary.

CASE 2: \( a_1 \neq 0 \), then

\[
\alpha = \frac{1}{392}(39 + 38\beta + 8\beta^2)
\]

and

\[
a_2 = -\frac{168}{\gamma(3 + 2\beta)} c_1^2
\]

provided \( \beta \) is one of the roots of

\[
(104\beta^2 + 886\beta + 1487)(520\beta^3 + 2158\beta^2 - 1103\beta - 8871) = 0
\]
Subcase 2-a: If $\beta^2 = -\frac{1}{104}(886\beta + 1487)$, then

$$\alpha = \frac{2\beta + 5}{26}$$

$$a_0 = -\frac{49c_1^2(9983 + 4378\beta)}{26\gamma(8 + 3\beta)(3 + 2\beta)^2}$$

$$a_1 = \pm\frac{336c_1^2}{\gamma(3 + 2\beta)}$$

$$a_2 = -\frac{168c_1^2}{\gamma(3 + 2\beta)}$$

$$c_2 = -\frac{364c_1^5(3851 + 1634\beta)}{6715 + 2946\beta}.\]$$

Subcase 2-b: If $\beta^3 = \frac{1}{520}(8871 + 1103\beta - 2158\beta^2)$, then

$$\alpha = \frac{39 + 38\beta + 8\beta^2}{392}$$

$$a_0 = \frac{28c_1^2(6483 + 5529\beta + 1066\beta^2)}{(3 + 2\beta)(23 + 6\beta)(81 + 26\beta)\gamma}$$

$$a_1 = \frac{28224c_1^4(4\beta - 1)(26\beta - 17)}{(3 + 2\beta)^2(23 + 6\beta)(81 + 26\beta)^2\gamma^2}$$

$$a_2 = -\frac{168c_1^2}{\gamma(3 + 2\beta)}$$

$$c_2 = -\frac{8c_1^5(1792261977 + 1161063881\beta + 188900114\beta^2)}{959833473 + 632954969\beta + 105176786\beta^2}.\]$$
Implementation Issues – Software Demo – Future Work

• Demonstration of Mathematica package for tanh/sech methods.

• Long term goal: Develop PDESolve for closed form solutions of nonlinear PDEs and DDEs.

• Implement various methods: Lie symmetry (similarity) methods.

• Look at other types of explicit solutions involving
  – hyperbolic functions sinh, cosh, tanh, ...
  – other special functions.
  – complex exponentials combined with sech or tanh.

• Example: Set of ODEs from quantum field theory

\[
\begin{align*}
    u_{xx} &= -u + u^3 + auv^2 \\
    v_{xx} &= bv + cv^3 + av(u^2 - 1).
\end{align*}
\]

Try solutions ($c_2 = 0$ for ODEs)

\[
    u_i(x, t) = \sum_{j=0}^{M_i} a_{ij} \tanh^j[c_1 x + c_2 t + \delta] + \sum_{j=0}^{N_i} b_{ij} \sech^{2j+1}[c_1 x + c_2 t + \delta].
\]

or

\[
    u_i(x, t) = \sum_{j=0}^{M_i} (\tilde{a}_{ij} + \tilde{b}_{ij} \sech[c_1 x + c_2 t + \delta]) \tanh^j[c_1 x + c_2 t + \delta].
\]

Solitary wave solutions:

\[
\begin{align*}
    u &= \pm \tanh[\sqrt{\frac{a^2 - c}{2(a - c)}} x + \delta] \\
    v &= \pm \sqrt{\frac{1 - a}{a - c}} \sech[\sqrt{\frac{a^2 - c}{2(a - c)}} x + \delta],
\end{align*}
\]

provided $b = \sqrt{\frac{a^2 - c}{2(a - c)}}$.  

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• Example: Nonlinear Schrödinger equation (focusing/defocusing):

\[ i u_t + u_{xx} \pm |u|^2 u = 0. \]

Bright soliton solution (+ sign):

\[ u(x, t) = \frac{k}{\sqrt{2}} \exp[i \left( \frac{c}{2} x + (k^2 - \frac{c^2}{4}) t \right)] \ \text{sech}[k(x - ct - x_0)] \]

Dark soliton solution (− sign):

\[ u(x, t) = \frac{1}{\sqrt{2}} \exp[i(K x - (2k^2 + 3K^2 - 2Kc + \frac{c^2}{2}) t)] \]
\[ \left\{ k \ \text{tanh}[k(x - ct - x_0)] - i(K - \frac{c}{2}) \right\}. \]

• Example: Nonlinear sine-Gordon equation (light cone coordinates):

\[ u_{xt} = \sin u. \]

Setting \( \Phi = u_x, \ \Psi = \cos(u) - 1 \), gives

\[ \Phi_{xt} - \Phi - \Phi \Psi = 0 \]
\[ 2\Psi + \Psi^2 + \Phi^2_t = 0. \]

Solitary wave solution (kink):

\[ \Phi = u_x = \pm \frac{1}{\sqrt{-c}} \ \text{sech}\left[\frac{1}{\sqrt{-c}}(x - ct) + \delta\right], \]
\[ \Psi = \cos(u) - 1 = 1 - 2 \ \text{sech}^2\left[\frac{1}{\sqrt{-c}}(x - ct) + \delta\right], \]

in final form:

\[ u(x, t) = \pm 4 \ \text{arctan}\left(\exp\left(\frac{1}{\sqrt{-c}}(x - ct) + \delta\right)\right). \]
• Example: Coupled nonlinear Schrödinger equations:

\[
\begin{align*}
    i u_t &= u_{xx} + u(|u|^2 + h|v|^2) \\
    i v_t &= v_{xx} + v(|v|^2 + h|u|^2)
\end{align*}
\]

Seek particular solutions

\[
\begin{align*}
    u(x, t) &= a \tanh(\mu x) \exp(iAt) \\
    v(x, t) &= b \text{sech}(\mu x) \exp(iBt).
\end{align*}
\]

• Seek solutions \( u(x, t) = U(F(\xi)) \), where derivatives of \( F(\xi) \) are polynomial in \( F \).

Now,

\[
F'(\xi) = 1 - F^2(\xi) \quad \rightarrow \quad F = \tanh(\xi).
\]

Other choices are possible.

• Add the constraining differential equations to the system of PDEs directly.

• Why are tanh and sech solutions so prevalent?

• Other applications:

    Computation of conservation laws, symmetries, first integrals, etc. leading to linear parameterized systems for unknowns coefficients (see InvariantsSymmetries by Göktaş and Hereman).