Symbolic Computation of Conserved Densities
Generalized Symmetries, and Recursion Operators for
Nonlinear Evolution and Lattice Equations

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Outline

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Part I  Purpose, Motivation, Strategy, Demo

• Purpose

Design and implement algorithms to compute polynomial conservation laws, generalized symmetries, and recursion operators for nonlinear systems of PDEs and differential-difference equations (DDEs).

• Motivation

– Conservation laws describe the conservation of fundamental physical quantities (linear momentum, energy, etc.).
  Compare with constants of motion (linear momentum, energy) in mechanics.

– Conservation laws provide a method to study quantitative and qualitative properties of equations and their solutions, e.g. Hamiltonian structures.

– Conserved densities can be used to test numerical integrators.

– For PDEs and DDEs, the existence of a sufficiently large (in principal infinite) number of conservation laws or symmetries assures complete integrability.

– Conserved densities and symmetries aid in finding the recursion operator (which guarantees the existence of infinitely many symmetries).
Definitions and Examples for PDEs

- **Nonlinear system of evolution equations**

  \[ u_t = F(u, u_x, u_{2x}, ..., u_{mx}) \]

  in a (single) space variable \( x \) and time \( t \).

  Notation:

  \[
  \begin{align*}
  u &= (u_1, u_2, ..., u_n), \\
  F &= (F_1, F_2, ..., F_n), \\
  u_t &= \frac{\partial u}{\partial t}, \\
  u_{ix} &= \frac{\partial u}{\partial x^i}.
  \end{align*}
  \]

  \( F \) is polynomial in \( u, u_x, u_{2x}, ..., u_{mx} \).

  PDEs of higher order in \( t \) should be recast as a first-order system.

- **Prototypical Examples**

  The Korteweg-de Vries (KdV) equation:

  \[ u_t + uu_x + u_{3x} = 0. \]

  Fifth-order evolution equations with constant parameters \((\alpha, \beta, \gamma)\):

  \[ u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma uu_{3x} + u_{5x} = 0. \]

  Special case. The fifth-order Sawada-Kotera (SK) equation:

  \[ u_t + 5u^2 u_x + 5u_x u_{2x} + 5uu_{3x} + u_{5x} = 0. \]

  The Boussinesq (wave) equation:

  \[ u_{tt} - u_{2x} + 3uu_{2x} + 3u_x^2 + \alpha u_{4x} = 0, \]

  written as a first-order system \((v\) auxiliary variable): \[
  \begin{align*}
  u_t + v_x &= 0, \\
  v_t + u_x - 3uu_x - \alpha u_{3x} &= 0.
  \end{align*}
  \]
A vector nonlinear Schrödinger equation (Verheest, Deconinck, Meuris):

\[ B_t + (|B|^2B)_x + (B_0 \cdot B_x)B_0 + e \times B_{xx} = 0, \]

written in components, \( B_0 = (a, b, 0), B = (u, v, 0), e = (0, 0, 1) : \)

\[ u_t + \left[ u(u^2 + v^2) + \beta u + \gamma v - v_x \right]_x = 0, \]
\[ v_t + \left[ v(u^2 + v^2) + \theta u + \delta v + u_x \right]_x = 0, \]

\( \beta = a^2, \gamma = \theta = ab, \) and \( \delta = b^2. \)

- **Dilation invariance of PDEs**

The KdV equation

\[ u_t + uu_x + u_{3x} = 0 \]

has scaling symmetry

\[ (t, x, u) \rightarrow (\lambda^{-3} t, \lambda^{-1} x, \lambda^2 u). \]

The Boussinesq system

\[ u_t + v_x = 0, \]
\[ v_t + u_x - 3uu_x - \alpha u_{3x} = 0, \]

is not scaling invariant (\( u_x \) and \( u_{3x} \) are conflicting terms).

If one introduces an auxiliary parameter \( \beta, \) then

\[ u_t + v_x = 0, \]
\[ v_t + \beta u_x - 3uu_x - \alpha u_{3x} = 0, \]

has scaling symmetry:

\[ (x, t, u, v, \beta) \rightarrow (\lambda^{-1} x, \lambda^{-2} t, \lambda^2 u, \lambda^3 v, \lambda^2 \beta). \]
• Computation of the dilation symmetry (Lie-point symmetry)

The **weight** $w$ of a variable is by definition the number of $x$-derivatives the variable corresponds to.

The **rank** of a monomial is its total weight in terms of $x$-derivatives.

**Example 1**: KdV Equation

Set $w(D_x) = 1$ or $w(x) = -1$ and require that all terms in

$$u_t + uu_x + u_{3x} = 0$$

have the same rank. Hence,

$$w(u) + w(D_t) = 2w(u) + 1 = w(u) + 3.$$

Solve the linear system

$$w(u) = 2, \ w(D_t) = 3, \ \text{or} \ w(t) = -3.$$

So,

$$(t, x, u) \rightarrow (\lambda^{-3}t, \lambda^{-1}x, \lambda^2u).$$

**Example 2**: Boussinesq System

$$u_t + v_x = 0,$$

$$v_t + \beta u_x - 3uu_x - \alpha u_{3x} = 0.$$

Solve

$$w(u) + w(D_t) = w(v) + 1,$$

$$w(v) + w(D_t) = w(\beta) + w(u) + 1 = 2w(u) + 1 = w(u) + 3$$

to get

$$w(u) = 2, \ w(v) = 3, \ w(\beta) = 2, \ w(D_t) = 2.$$

Hence,

$$(x, t, u, v, \beta) \rightarrow (\lambda^{-1}x, \lambda^{-2}t, \lambda^2u, \lambda^3v, \lambda^2\beta).$$
• Conservation Law for PDEs

\[ D_t \rho + D_x J = 0 \quad \text{on PDE}, \]

conserved density \( \rho \) and flux \( J \).

Both are polynomial in \( u, u_x, u_{2x}, u_{3x}, \ldots \)

\[ P = \int_{-\infty}^{+\infty} \rho \, dx = \text{constant in time} \]

if \( J \) vanishes at infinity.

Examples

Conserved densities and fluxes for the Korteweg-de Vries (KdV) equation

\[ u_t + uu_x + u_{3x} = 0 \]

\[ \rho_{(1)} = u, \quad D_t(u) + D_x\left(\frac{u^2}{2} + u_{2x}\right) = 0. \]

\[ \rho_{(2)} = u^2, \quad D_t(u^2) + D_x\left(\frac{2u^3}{3} + 2uu_{2x} - u_x^2\right) = 0. \]

\[ \rho_{(3)} = u^3 - 3u_x^2, \quad D_t\left(u^3 - 3u_x^2\right) + D_x\left(\frac{3}{4}u^4 - 6uu_x^2 + 3u^2u_{2x} + 3u_{2x}^2 - 6u_xu_{3x}\right) = 0. \]

\[ \vdots \]

\[ \rho_{(6)} = u^6 - 60u^3u_x^2 - 30u_x^4 + 108u^2u_{2x}^2 + \frac{720}{7}u_{2x}^3 - \frac{648}{7}uu_{3x}^2 + \frac{216}{7}u_{4x}^2. \]

The Boussinesq system (before setting \( \beta = 1 \)):

\[ \rho_{(1)} = u, \quad \rho_{(2)} = v, \]

\[ \rho_{(3)} = uv, \quad \rho_{(4)} = \beta u^2 - u^3 + v^2 + \alpha u_x^2. \]
• Generalized Symmetries of PDEs.

\[ G(x, t, u, u_x, u_{2x}, \ldots) \]

with \( G = (G_1, G_2, \ldots, G_n) \) is a *symmetry* iff it leaves the PDE invariant for the replacement \( u \to u + \epsilon G \) within order \( \epsilon \). i.e.

\[ D_t(u + \epsilon G) = F(u + \epsilon G) \]

must hold up to order \( \epsilon \) on the solutions of PDE.

Consequently, \( G \) must satisfy the linearized equation

\[
D_t G = F'(u)[G] = \frac{\partial}{\partial \epsilon} F(u + \epsilon G)|_{\epsilon=0} = \sum_{i=0}^m (D^i_x G) \frac{\partial F}{\partial u_{ix}}
\]

where \( m \) is the highest order and \( F' \) is the Fréchet derivative of \( F \) in the direction of \( G \).

Here, \( u \) is replaced by \( u + \epsilon G \), and \( u_{ix} \) by \( u_{ix} + \epsilon D^i_x G \).

**Example**

For the KdV equation, \( u_t = 6uu_x + u_{3x} \), the first few generalized symmetries are:

\[ G^{(1)} = u_x, \]
\[ G^{(2)} = 6uu_x + u_{3x}, \]
\[ G^{(3)} = 30u^2 u_x + 20u_x u_{2x} + 10 uu_{3x} + u_{5x}, \]
\[ G^{(4)} = 140u^3 u_x + 70u_x^3 + 280 uu_x u_{2x} + 70u^2 u_{3x} \]
\[ + 70 u_{2x} u_{3x} + 42u_x u_{4x} + 14 uu u_{5x} + u_{7x}. \]
- Recursion Operators of PDEs.

A recursion operator $\mathcal{R}$ connects symmetries

$$G^{(j+s)} = \mathcal{R}G^{(j)}, \ j = 1, 2, \ldots$$

$s$ is seed ($s = 1$ in simplest case).

For $n$-component systems, $\mathcal{R}$ is an $n \times n$ matrix.

Defining equation for $\mathcal{R}$:

$$D_t \mathcal{R} + [\mathcal{R}, F'(u)] = \frac{\partial \mathcal{R}}{\partial t} + \mathcal{R}'[F] + \mathcal{R} \circ F'(u) - F'(u) \circ \mathcal{R} = 0,$$

where $[\ , \ ]$ means commutator, $\circ$ stands for composition, and

$$F'(u) = D_F = \sum_{i=0}^{m} \left( \frac{\partial F}{\partial u_{ix}} \right) D_x^i$$

$m$ is highest order, $D_x$ is differential operator and $D_x^i = D_x \circ D_x \circ \cdots \circ D_x$ ($i$ times).

$\mathcal{R}'[F]$ is the Fréchet derivative of $\mathcal{R}$ in direction of $F$:

$$\mathcal{R}'[F] = \sum_{i=0}^{n} (D_x^i F) \frac{\partial \mathcal{R}}{\partial u_{ix}}$$

Example

The recursion operator for the KdV equation:

$$\mathcal{R} = D_x^2 + 2uI + 2D_x u D_x^{-1} = D_x^2 + 4uI + 2u_x D_x^{-1}.$$ 

$D_x$ is differentiation and $D_x^{-1}$ is integration operator.

For example,

$$\mathcal{R}u_x = (D_x^2 + 2uI + 2D_x u D_x^{-1})u_x = 6uu_x + u_{3x},$$

$$\mathcal{R}(6uu_x + u_{3x}) = (D_x^2 + 2uI + 2D_x u D_x^{-1})(6uu_x + u_{3x})$$

$$= 30u^2u_x + 20u_x u_{2x} + 10uu_{3x} + u_{5x}.$$
Example: The modified KdV equation.

The modified Korteweg-de Vries (mKdV) equation:

\[ u_t = u^2 u_x + u_{3x} \]

Scaling invariance:

\[(t, x, u) \rightarrow (\lambda^{-3}t, \lambda^{-1}x, \lambda^1u)\]

First three conservation laws:

\[ D_t(u) + D_x(-\frac{1}{3}u^3 - u_{2x}) = 0 \]

\[ D_t(u^2) + D_x(-\frac{1}{2}u^4 + u_x^2 - 2uu_{2x}) = 0 \]

\[ D_t(u^4 - 6u_x^2) + D_x(-\frac{2}{3}u^6 + 12u^2u_x^2 - 4u^3u_{2x} - 6u_{2x}^2 + 12u_xu_{3x}) = 0 \]

First three generalized symmetries:

\[ G^{(1)} = u_x \]

\[ G^{(2)} = u^2u_x + u_{3x} \]

\[ G^{(3)} = \frac{5}{6}u_x^4 + \frac{20}{3}u_xu_{2x} + \frac{5}{3}u^2u_{3x} + \frac{5}{3}u_x^3 + u_{5x} \]

Recursion operator:

\[ \mathcal{R} = D_x^2 + \frac{2}{3}u^2I + \frac{2}{3}u_xD_x^{-1}uI \]
• Example: Dispersionless long wave system.

Dispersionless long wave system:

\[
\mathbf{u}_t = \begin{pmatrix} u_t \\ v_t \end{pmatrix} = \mathbf{F} = \begin{pmatrix} uv_x + u_x v \\ u_x + vv_x \end{pmatrix}
\]

Conserved densities:

\[
\begin{align*}
\rho(1) &= v \\
\rho(2) &= u \\
\rho(3) &= uv \\
\rho(4) &= u^2 + uv^2
\end{align*}
\]

Generalized symmetries:

\[
\begin{align*}
\mathbf{G}^{(1)} &= \begin{pmatrix} u_x \\ v_x \end{pmatrix} \\
\mathbf{G}^{(2)} &= \begin{pmatrix} uv_x + u_x v \\ u_x + vv_x \end{pmatrix} \\
\mathbf{G}^{(3)} &= \begin{pmatrix} 2uu_x + 2uvv_x + u_xv^2 \\ 2uv_x + 2u_xv + v^2v_x \end{pmatrix}
\end{align*}
\]

Recursion operator:

\[
\mathcal{R} = \begin{pmatrix} vI & 2uI + u_xD_x^{-1} \\ 2I & vI + v_xD_x^{-1} \end{pmatrix}
\]

Note:

\[
\text{rank}(\mathcal{R}) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}
\]
Definitions and Examples for DDEs (lattices)

- **Nonlinear system of DDEs**
  (continuous in time, discretized in space)
  \[
  \dot{u}_n = F(..., u_{n-1}, u_n, u_{n+1}, ...),
  \]
  \(u_n\) and \(F\) are vector dynamical variables.
  \(F\) is polynomial with constant coefficients (parameters).
  No restrictions on the level of the shifts or the degree of nonlinearity.

- **Example**
  One-dimensional Toda lattice
  \[
  \ddot{y}_n = \exp (y_{n-1} - y_n) - \exp (y_n - y_{n+1}).
  \]
  \(y_n\) is the displacement from equilibrium of the \(n\)th particle with unit mass under an exponentially decaying interaction force between nearest neighbors.
  Change of variables:
  \[
  u_n = \dot{y}_n, \quad v_n = \exp (y_n - y_{n+1})
  \]
  yields
  \[
  \dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n (u_n - u_{n+1}).
  \]
  Toda system is completely integrable.
• Dilation Invariance of DDEs

The Toda lattice

\[ \dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}). \]

is invariant under the scaling symmetry

\[ (t, u_n, v_n) \rightarrow (\lambda^{-1}t, \lambda u_n, \lambda^2 v_n). \]

Weights \( w(u_n), w(v_n) \) are defined in terms of \( t \)-derivatives.

Using \( w\left(\frac{d}{dt}\right) = 1, \ w(u_{n\pm p}) = w(u_n), \ w(v_{n\pm p}) = w(v_n) \)

\[ w(u_n) + 1 = w(v_n), \]

\[ w(v_n) + 1 = w(v_n) + w(u_n). \]

Hence,

\[ w(u_n) = 1, \ w(v_n) = 2. \]

The rank of a monomial is its total weight in terms of \( t \)-derivatives.

• Conservation Law for DDEs:

\[ \dot{\rho}_n = J_n - J_{n+1} \quad \text{on DDE,} \]

density \( \rho_n \), flux \( J_n \).

\[ \frac{d}{dt}(\sum_n \rho_n) = \sum_n \dot{\rho}_n = \sum_n (J_n - J_{n+1}) \]

if \( J_n \) is bounded for all \( n \).

Subject to suitable boundary or periodicity conditions

\[ \sum_n \rho_n = \text{constant}. \]

The first three density-flux pairs (computed by hand):

\[ \rho_n^{(0)} = \ln(v_n) \quad J_n^{(0)} = u_n \]

\[ \rho_n^{(1)} = u_n \quad J_n^{(1)} = v_{n-1} \]

\[ \rho_n^{(2)} = \frac{1}{2} u_n^2 + v_n \quad J_n^{(2)} = u_n v_{n-1} \]
• Generalized Symmetries of DDEs

A vector function \( G(\ldots, u_{n-1}, u_n, u_{n+1}, \ldots) \) is a \textit{symmetry} iff the infinitesimal transformation

\[
u_n \rightarrow u_n + \epsilon G(\ldots, u_{n-1}, u_n, u_{n+1}, \ldots)
\]

leaves the DDE system invariant within order \( \epsilon \).

\( G \) must satisfy the linearized equation

\[
D_t G = F'(u_n)[G] = \frac{\partial}{\partial \epsilon} F(u_n + \epsilon G)|_{\epsilon=0} = \sum_{i=-q}^{p} (D^i G) \frac{\partial F}{\partial u_{n+i}},
\]

where \( F' \) is the Fréchet derivative of \( F \) in direction of \( G \).

\( D \) is \textbf{up-shift operator}, \( D^{-1} \) is \textbf{down-shift operator}, and \( D^i = D \circ D \circ \cdots \circ D \) (\( i \) times).

Here,

\[
u_n \rightarrow u_n + \epsilon G(\ldots, u_{n-1}, u_n, u_{n+1}, \ldots),
\]

\[
u_{n\pm k} \rightarrow u_{n\pm k} + \epsilon G|_{n\rightarrow n\pm k}.
\]

• Example

Consider the Toda lattice

\[
\dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}).
\]

First three higher-order symmetries:

\[
G^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

\[
G^{(2)} = \begin{pmatrix} v_n - v_{n-1} \\ v_n(u_n - u_{n+1}) \end{pmatrix}
\]

\[
G^{(3)} = \begin{pmatrix} v_n(u_n + u_{n+1}) - v_{n-1}(u_{n-1} + u_n) \\ v_n(u^2_{n+1} - u^2_n + v_{n+1} - v_{n-1}) \end{pmatrix}
\]
• Recursion Operators of DDEs.

A recursion operator $R$ connects symmetries

$$G^{(j+s)} = R G^{(j)} , \ j = 1, 2, \ldots,$$

$s$ is seed. For $r$-component systems, $R$ is an $r \times r$ matrix.

Defining equation for $R$ :

$$D_t R + [R, F'(u_n)] = \frac{\partial R}{\partial t} + R'[F] + R \circ F'(u_n) - F'(u_n) \circ R = 0,$$

where $[,]$ means commutator, $\circ$ stands for composition, and

$$F'(u_n) = \sum_{i=-q}^{p} \left( \frac{\partial F}{\partial u_{n+i}} \right) D^i$$

$p, q$ are bounds of the shifts, $D$ is up-shift operator and $D^i = D \circ D \circ \cdots \circ D$ ($i$ times).

$R'[F]$ is the Fréchet derivative of $R$ in direction of $F$ :

$$R'[F] = \sum_{i=-q}^{p} \left( D^i F \right) \frac{\partial R}{\partial u_{n+i}}$$

**Example 1**

The Kac-van Moerbeke equation

$$\dot{u}_n = u_n(u_{n+1} - u_{n-1}),$$

has recursion operator

$$R = u_n D + u_n D^{-1} + (u_n + u_{n+1}) I + u_n(u_{n+1} - u_{n-1})(D - I)^{-1} \frac{1}{u_n} I$$

$$= u_n(I + D)(u_n D - D^{-1} u_n)(D - I)^{-1} \frac{1}{u_n} I$$

Note: $\rho_n^{(0)} = \ln(u_n)$ and $J_n^{(0)} = -(u_n + u_{n-1})$ are density-flux pair.
Example 2

The (quadratic) Volterra equation

\[ \dot{u}_n = u_n^2(u_{n+1} - u_{n-1}) \]

has recursion operator

\[ \mathcal{R} = u_n^2 D + u_n^2 D^{-1} + 2u_n u_{n+1} I + 2u_n^2 (u_{n+1} - u_{n-1})(D - I)^{-1} \frac{1}{u_n} I \]

Example 3

The Toda lattice

\[ \dot{u}_n = v_{n-1} - v_n \quad \dot{v}_n = v_n (u_n - u_{n+1}) \]

has recursion operator

\[ \mathcal{R} = \begin{pmatrix} -u_n I & -D^{-1} - I + (v_{n-1} - v_n)(D - I)^{-1} \frac{1}{v_n} I \\ -vI - vD & u_{n+1} I + v_n (u_n - u_{n+1})(D - I)^{-1} \frac{1}{v_n} I \end{pmatrix} \]

The recursion operator can be factored as

\[ \mathcal{R} = \mathcal{H}\mathcal{S} \]

with Hamiltonian (symplectic) operator

\[ \mathcal{H} = \begin{pmatrix} D^{-1}v_n I - v_n D & -u_n v_n I + u_n D^{-1} v_n I \\ -v_n D u_n I + u_n v_n I & -v_n D v_n I + v_n D^{-1} v_n I \end{pmatrix} \]

and co-symplectic operator

\[ \mathcal{S} = \begin{pmatrix} 0 & (D - I)^{-1} \frac{1}{v_n} I \\ \frac{1}{v_n} D (D - I)^{-1} & 0 \end{pmatrix} \]
• **Key Observation**

Conserved densities, generalized symmetries, and recursion operators are invariant under the dilation (scaling) symmetry of the given PDE or DDE.

• **Overall Strategy**

   **Exploit dilation symmetry as much as possible.**

   **Keep the computations as simple as possible.**

   **Use linear algebra**
   * solve linear systems
   * construct basis vectors (building blocks)
   * use linear independence
   * work in finite dimensional spaces

   **Use calculus and differential equations**
   * derivatives
   * integrals (as little as possible)
   * solve systems of linear ODEs

   **Use tools from variational calculus**
   * variational derivative (Euler operator)
   * Fréchet derivative
   * calculus with operators

   **Use analogy between continuous and semi-discrete cases**
Analogy PDEs and DDEs

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<th>Continuous Case (PDEs)</th>
<th>Semi-discrete Case (DDEs)</th>
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<td>$u_t = F(u, u_x, u_{2x}, ...)$</td>
<td>$\dot{u}<em>n = F(..., u</em>{n-1}, u_n, u_{n+1}, ...)$</td>
<td></td>
</tr>
<tr>
<td>Conservation Law</td>
<td>$D_t \rho + D_x J = 0$</td>
<td>$\dot{\rho}<em>n + J</em>{n+1} - J_n = 0$</td>
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<tr>
<td>Symmetry</td>
<td>$D_t G = F'(u)[G] = \frac{\partial}{\partial \epsilon} F(u + \epsilon G)</td>
<td>_{\epsilon=0}$</td>
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<tr>
<td>Recursion Operator</td>
<td>$D_t \mathcal{R} + [\mathcal{R}, F'(u)] = 0$</td>
<td>$D_t \mathcal{R} + [\mathcal{R}, F'(u_n)] = 0$</td>
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Table 1: Conservation Laws and Symmetries

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<th>Equation</th>
<th>KdV Equation</th>
<th>$u_t = 6uu_x + u_{3x}$</th>
<th>$\dot{u}<em>n = u_n (u</em>{n+1} - u_{n-1})$</th>
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<td>Densities</td>
<td>$\rho = u$, $\rho = u^2$</td>
<td>$\rho_n = u_n$, $\rho_n = u_n (\frac{1}{2} u_n + u_{n+1})$</td>
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<tr>
<td>$\rho = u^3 - \frac{1}{2} u_x^2$</td>
<td>$\rho_n = \frac{1}{3} u_{n+1} + u_n u_{n+1} (u_{n+1} + u_{n+2})$</td>
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<td>Symmetries</td>
<td>$G = u_x$, $G = 6uu_x + u_{3x}$</td>
<td>$G = u_n u_{n+1} (u_n + u_{n+1} + u_{n+2})$</td>
<td></td>
</tr>
<tr>
<td>$G = 30u^2u_x + 20u_x u_{2x}$</td>
<td>$-u_{n-1} u_n (u_{n-2} + u_{n-1} + u_{n})$</td>
<td>$-u_{n-1} u_n (u_{n-2} + u_{n-1} + u_{n})$</td>
<td></td>
</tr>
<tr>
<td>$+10u u_{3x} + u_{5x}$</td>
<td>Recursion Operator</td>
<td>$\mathcal{R} = D_x^2 + 4u + 2u_x D^{-1}$</td>
<td>$\mathcal{R} = u_n (1 + D)(u_n D - D^{-1} u_n)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$(D - 1)^{-1} \frac{1}{u_n}$</td>
</tr>
</tbody>
</table>

Table 2: Prototypical Examples
Part II  Algorithms for PDEs

- Algorithm for Conserved Densities of PDEs.

(i) Determine weights (scaling properties) of variables and auxiliary parameters.
(ii) Construct the form of the density (find monomial building blocks).
(iii) Determine the constant coefficients (parameters).

Example: Density of rank 6 for the KdV equation

\[ u_t + uu_x + u_{3x} = 0 \]

Step 1: Compute the weights (dilation symmetry).

Solve

\[ w(u) + w(D_t) = 2w(u) + 1 = w(u) + 3. \]

Hence,

\[ w(u) = 2, \quad w(D_t) = 3. \]

Step 2: Determine the form of the density.

List all possible powers of \( u \), up to rank 6: \([u, u^2, u^3]\).

Introduce \( x \) derivatives to ‘complete’ the rank.

- \( u \) has weight 2, introduce \( D_x^4 \).
- \( u^2 \) has weight 4, introduce \( D_x^2 \).
- \( u^3 \) has weight 6, no derivative needed.
Apply the $D_x$ derivatives.

Remove total derivative terms ($D_x u_{px}$) and highest derivative terms as follows:

\[
[u_{4x}] \rightarrow [\ ] \quad \text{empty list.}
\]
\[
[u_x^2, uu_{2x}] \rightarrow [u_x^2] \quad \text{since } uu_{2x} = (uu_x)_x - u_x^2.
\]
\[
[u^3] \rightarrow [u^3].
\]

Linearly combine the ‘building blocks’:

\[
\rho = c_1 u^3 + c_2 u_x^2.
\]

**Step 3: Determine the coefficients $c_i$.**

Use the defining equation

\[
D_t \rho + D_x J = 0.
\]

Compute

\[
D_t \rho = \frac{\partial \rho}{\partial t} + \sum_{k=0}^{m} \frac{\partial \rho}{\partial u_{kx}} D_x^k u_t
\]
\[
= \frac{\partial \rho}{\partial t} + \rho'(u)[F].
\]

since $u_t = F$. Here, $D_t \rho = 3c_1 u^2 u_t + 2c_2 u_x u_{xt}$.

Replace $u_t$ by $-(uu_x + u_{3x})$ and $u_{xt}$ by $-(uu_x + u_{3x})_x$.

Hence,

\[
E = D_t \rho = -3c_1 u^2 (uu_x + u_{3x}) - 2c_2 u_x (uu_x + u_{3x})_x.
\]

Apply the Euler operator (variational derivative)

\[
\mathcal{L}_u = \sum_{k=0}^{m} (-1)^k D_x^k \frac{\partial}{\partial u_{kx}}
\]
\[
= \frac{\partial}{\partial u} - D_x \left( \frac{\partial}{\partial u_x} \right) + D_x^2 \left( \frac{\partial}{\partial u_{2x}} \right) + \cdots + (-1)^m D_x^m \left( \frac{\partial}{\partial u_{mx}} \right).
\]
to $E$ of order $m = 4$. Result:

$$\mathcal{L}_u(E) = -(3c_1 + c_2)u_x^3.$$ 

This non-integrable term must vanish.

So, $c_1 = -\frac{1}{3}c_2$. Set $c_2 = -3$, then $c_1 = 1$.

Hence,

$$\rho = u^3 - 3u_x^2.$$ 

Integration of $E = -D_xJ$ yields

$$J = \frac{3}{4}u^4 - 6uu_x^2 + 3u^2u_{2x} + 3u_{2x}^2 - 6u_xu_{3x}.$$ 

Note: Rank $J = \text{Rank } \rho + \text{Rank } D_t - 1$.

If integration by parts fails: Build up form of $J$.

Compute

$$D_xJ = \frac{\partial J}{\partial x} + \sum_{k=0}^{m} \frac{\partial J}{\partial u_{kx}}u_{(k+1)x},$$

($m$ is the order of $J$) and match $D_xJ = -E$. 

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Single Mixed-Derivative Equations (PDEs) with Transcendental Nonlinearities

- **Nonlinear evolution equation**
  
  \[ u_{xt} = F(u, u_x, u_{2x}, \ldots, u_{mx}), \]

  scalar \( u(x, t) \) and \( F \). But \( F \) is transcendental in \( u \) but polynomial in \( u_x, \ldots, u_{mx} \).

- **Examples:**
  The sine-Gordon (and sinh-Gordon) equations:
  
  \[ u_{xt} = \alpha \sin(u) \quad \text{and} \quad u_{x,t} = \alpha \sinh(u). \]

  The Liouville equation:
  
  \[ u_{xt} = \alpha \exp(u). \]

- **Dilation Invariance**
  The sine-Gordon and Liouville equations have scaling symmetry
  
  \[ (t, x, u, \alpha) \to (\lambda^{-1}t, \lambda^{-1}x, u, \lambda^2\alpha). \]

  Here, \( w(u) = 0, \ w(D_x) = 1, \ w(D_t) = 1, \) and \( w(\alpha) = 2. \)

  The sine-Gordon and Liouville equations are uniform of rank 2.
  (if \( \alpha \) is introduced).

  Without \( \alpha \), the equations would be uniform of rank 0, and \( w(D_t) = -1. \)

  Use densities algorithm on variable \( \tilde{u} = u_x \) instead of \( u. \)
• Conserved densities for sine-Gordon equation

Rank 2:

\[ \rho_1 = u_x^2, \quad J_1 = 2\alpha \cos(u) \]

Rank 4:

\[ \rho_2 = u_x^4 - 4u_x^2, \quad J_2 = 4\alpha \cos(u)u_x^2. \]

Rank 6:

\[ \rho_3 = u_x^6 - 20u_x^2u_x^2 + 8u_x^2, \]

\[ J_3 = 6\alpha \cos(u)u_x^4 + 16\alpha \sin(u)u_x^2u_x - 8\alpha \cos(u)u_x^2. \]

Rank 8:

\[ \rho_4 = u_x^8 - 56u_x^4u_x^2 - \frac{112}{5}u_x^4 + \frac{224}{5}u_x^2u_x^2 - \frac{64}{5}u_x^2, \]

\[ J_4 = 8\alpha \cos(u)u_x^6 + 64\alpha \sin(u)u_x^4u_x + 32\alpha \cos(u)u_x^2u_x^2 \]

\[ + \frac{128}{5}\alpha \sin(u)u_x^3 - \frac{128}{5}\alpha \cos(u)u_x^3u_x - \frac{384}{5}\alpha \sin(u)u_xu_x^2u_x \]

\[ + \frac{64}{5}\alpha \cos(u)u_x^2. \]

• Conserved densities for Liouville equation

Similar results as for sine-Gordon equation.

Conserved density with time and space dependent coefficients:

\[ \rho = f'(x)u_x + \frac{1}{2}f(x)u_x^2, \quad J = -f(x) \exp(u). \]
Systems of Evolution Equations (PDEs) with Transcendental Nonlinearities

- Nonlinear system of evolution equations

\[ u_t = F(u, u_x, u_{2x}, \ldots, u_{mx}) \]

in a (single) space variable \( x \) and time \( t \),

\[ u = (u_1, u_2, \ldots, u_n), \quad F = (F_1, F_2, \ldots, F_n). \]

\( F \) is transcendental in \( u \) and polynomial in \( u_x, \ldots, u_{mx} \).

- Prototypical Example:

The sine-Gordon system:

\[
\begin{align*}
    u_t &= v, \\
    v_t &= u_{xx} + \alpha \sin(u).
\end{align*}
\]

- Dilation Invariance

The sine-Gordon system has scaling symmetry

\[(t, x, u, v, \alpha) \rightarrow (\lambda^{-1}t, \lambda^{-1}x, u, \lambda v, \lambda^2 \alpha).\]

Here, \( w(u) = 0, \ w(v) = 1, \ w(D_x) = 1, \ w(D_t) = 1, \) and \( w(\alpha) = 2. \)

First equation has rank 1, the second has rank 2.
Without parameter \( \alpha \), the system would not be uniform in rank.
• **Example:** Density of rank 2 for the sine-Gordon system

\[
\begin{align*}
    u_t &= v, \\
    v_t &= u_{xx} + \alpha \sin(u).
\end{align*}
\]

**Step 1: Compute the weights:**

\[ w(u) = 0, \quad w(v) = 1, \quad w(\alpha) = 2. \]

**Step 2: Construct the form of the density.**

\[
\rho = \alpha h_1(u) + h_2(u)v^2 + h_3(u)u_x^2 + h_4(u)u_xv,
\]

where \( h_i(u) \) are unknown functions.

**Step 3: Determine the functions \( h_i \).**

Solve the system of linear ODEs:

\[
\begin{align*}
    h_2(u) - h_3(u) &= 0, \\
    h_2'(u) &= 0, \\
    h_3'(u) &= 0, \\
    h_4'(u) &= 0, \\
    h_2''(u) &= 0, \\
    h_4''(u) &= 0, \\
    2h_2'(u) - h_3'(u) &= 0, \\
    2h_2''(u) - h_3''(u) &= 0, \\
    h_1'(u) + 2\sin(u)h_2(u) &= 0, \\
    h_1''(u) + 2\sin(u)h_2'(u) + 2\cos(u)h_2(u) &= 0,
\end{align*}
\]

**Solution:**

\[
\begin{align*}
    h_1(u) &= 2c_1 \cos(u) + c_3, \\
    h_2(u) &= h_3(u) = c_1, \\
    h_4(u) &= c_2.
\end{align*}
\]
After substitution in $\rho$

$$\rho = 2c_1 \alpha \cos(u) + \alpha c_3 + c_1 v^2 + c_1 u_x^2 + c_2 u_x v.$$  

After splitting

$$\rho(0) = \alpha c_3,$$

$$\rho(1) = 2\alpha \cos(u) + v^2 + u_x^2,$$

$$\rho(2) = u_x v.$$  

**Conserved densities for sine-Gordon system**

Rank 2 (two conserved densities):

$$\rho(1) = 2\alpha \cos(u) + v^2 + u_x^2, \quad J(1) = -2u_x v,$$

$$\rho(2) = u_x v, \quad J(2) = -\left(\frac{1}{2} v^2 + \frac{1}{2} u_x^2 - \alpha \cos(u)\right).$$

Rank 4 (two new conserved densities) from

$$\tilde{\rho}_3 = c_1 [12\alpha \cos(u)v u_x + 2v^3 u_x + 2vu_x^3 - 16v_x u_{2x}]$$

$$+ c_2 [\alpha^2 (2 + 2\cos^2(u) - 2\sin^2(u)) + v^4 + 6v^2 u_x^2$$

$$+ u_x^4 + \alpha(4 \cos(u)v^2 + 20 \cos(u)u_x^2) - 16v_x^2 - 16u_{2x}^2]$$

$$+ c_3 [-16\alpha vu_x]$$

$$+ c_4 [-32\alpha^2 \cos(u) + \alpha(-16v^2 - 16u_x^2)]$$

$$+ c_5 [-16\alpha^2]$$

$J(3)$ is not shown (long).

After splitting and setting $\alpha = 1$:

$$\rho(3) = 12 \cos(u)v u_x + 2v^3 u_x + 2vu_x^3 - 16v_x u_{2x},$$

$$\rho(4) = 2\cos^2(u) - 2\sin^2(u) + v^4 + 6v^2 u_x^2 + u_x^4 + 4 \cos(u)v^2$$

$$+ 20 \cos(u)u_x^2 - 16v_x^2 - 16u_{2x}^2.$$
Algorithm for Generalized Symmetries of PDEs.

Consider the KdV equation, \( u_t = 6uu_x + u_{3x} \), with \( w(u) = 2 \).

Step 1: Construct the form of the symmetry.

Compute the form of the symmetry with rank 7.
List all powers in \( u \) with rank 7 or less:
\[
\mathcal{L} = \{1, u, u^2, u^3\}.
\]
For each monomial in \( \mathcal{L} \), introduce the needed \( x \)-derivatives, so that each term exactly has rank 7. Thus,
\[
\begin{align*}
D_x(u^3) & = 3u^2u_x, \\
D_x^3(u^2) & = 6u_xu_{2x} + 2uu_{3x}, \\
D_x^5(u) & = u_{5x}, \\
D_x^7(1) & = 0.
\end{align*}
\]
Gather the resulting (non-zero) terms
\[
\mathcal{R} = \{u^2u_x, u_xu_{2x}, uu_{3x}, u_{5x}\}.
\]
The symmetry is a linear combination of these monomials:
\[
G = c_1 u^2u_x + c_2 u_xu_{2x} + c_3 uu_{3x} + c_4 u_{5x}.
\]

Step 2: Determine the unknown coefficients \( c_i \).

Compute \( D_tG \) and use KdV to remove \( u_t, u_{tx}, u_{txx} \), etc.
Compute the Fréchet derivative \( F'(u)[G] \).
Equate the resulting expressions.
Group the terms:
\[
(12c_1 - 18c_2)u_x^2u_{2x} + (6c_1 - 18c_3)uu_x^2 + (6c_1 - 18c_3)uu_xu_{3x} + \\
(3c_2 - 60c_4)u_{3x}^2 + (3c_2 + 3c_3 - 90c_4)u_{2x}u_{4x} + (3c_3 - 30c_4)u_xu_{5x} \equiv 0.
\]
Solve the linear system:

\[
S = \{12c_1 - 18c_2 = 0, 6c_1 - 18c_3 = 0, 3c_2 - 60c_4 = 0, \\
3c_2 + 3c_3 - 90c_4 = 0, 3c_3 - 30c_4 = 0\}.
\]

Solution: \( \frac{c_1}{30} = \frac{c_2}{20} = \frac{c_3}{10} = c_4 \).

Setting \( c_4 = 1 \) one gets: \( c_1 = 30, c_2 = 20, c_3 = 10 \).

Substitute the result into the symmetry:

\[
G = 30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x}.
\]

Note: \( u_t = G \) is known as the Lax equation.

- **x-t Dependent symmetries.**

The KdV equation has also symmetries which explicitly depend on \( x \) and \( t \).

The same algorithm can be used provided the highest degree of \( x \) or \( t \) is specified.

Compute the symmetry of rank 2, that is linear in \( x \) or \( t \).

List all monomials in \( u, x \) and \( t \) of rank 2 or less:

\[
\mathcal{L} = \{1, u, x, xu, t, tu, tu^2\}.
\]

For each monomial in \( \mathcal{L} \), introduce enough \( x \)-derivatives, so that each term exactly has rank 2. Thus,

\[
D_x(xu) = u + xu_x, \quad D_x(tu^2) = 2tuux, \quad D_x^3(tu) = tu_{3x}, \\
D_x^2(1) = D_x^3(x) = D_x^5(t) = 0.
\]

Gather the non-zero resulting terms:

\[
\mathcal{R} = \{u, xu_x, tuux, tu_{3x}\},
\]
Build the linear combination

\[ G = c_1 u + c_2 xu_x + c_3 t uu_x + c_4 t u_{3x}. \]

Determine the coefficients \( c_1 \) through \( c_4 \):

\[ G = \frac{2}{3} u + \frac{1}{3} xu_x + 6t uu_x + tu_{3x}. \]

Two symmetries of KdV that explicitly depend on \( x \) and \( t \):

\[ G = 1 + 6tu_x, \quad \text{and} \quad G = 2u + xu_x + 3t(6uu_x + u_{3x}), \]

of rank 0 and 2, respectively.
Recursion Operators for PDEs

• Key Observation

★ The recursion operator of the KdV equation

\[ R = D_x^2 + 2u I + 2D_x u D_x^{-1} = D_x^2 + 4u I + 2u_x D_x^{-1} \]

has rank \( R = 2 \). \( D_x \) is differentiation and \( D_x^{-1} \) is integration operator.

Indeed, compare the ranks of the symmetries

\[ Ru_x = (D_x^2 + 2u I + 2D_x u D_x^{-1})u_x = 6uu_x + u_{3x}, \]
\[ R(6uu_x + u_{3x}) = (D_x^2 + 2u I + 2D_x u D_x^{-1})(6uu_x + u_{3x}) \]
\[ = 30u^2u_x + 20u_x u_{2x} + 10uu_{3x} + u_{5x}. \]

★ The terms in the recursion operator are monomials in \( D_x, D_x^{-1}, u, u_x, ... \)

★ Recursion operators split naturally in \( R = R_0 + R_1 \).

\( R_0 \) is a differential operator (no \( D_x^{-1} \) terms).
\( R_1 \) is an integral operator (with \( D_x^{-1} \) terms).

★ Application of \( R \) to a symmetry should not leave integrals.

For the KdV equation:

\( D_x^{-1}(6uu_x + u_{3x}) = 3u^2 + u_{2x} \) is polynomial.

Use the conserved densities: \( \rho(1) = u, \rho(2) = u^2, \rho(3) = u^3 - \frac{1}{2} u_x^2 \)

\[ D_t \rho(1) = D_t u = u_t = -D_x J(1), \]
\[ D_t \rho(2) = D_t u^2 = 2uu_t = -D_x J(2), \] and
\[ D_t \rho(3) = D_t (u^3 - \frac{1}{2} u_x^2) = \rho_3'(u)[u_t] = (3u^2 - u_x D_x)u_t = -D_x J(3), \]

for polynomial \( J(i), i = 1, 2, 3. \)

So, application of \( D_x^{-1} \), or \( D_x^{-1} u, \) or \( D_x^{-1}(3u^2 - u_x D_x) \)
to \( 6uu_x + u_{3x} \) leads to a polynomial result.
• Algorithm for Recursion Operators of PDEs.

Scalar Case

Step 1: Determine the rank of the recursion operator.

Recall: symmetries for the KdV equation, \( u_t = 6uu_x + u_{3x} \), are
\[
G^{(1)} = u_x, \quad G^{(2)} = 6uu_x + u_{3x},
\]
\[
G^{(3)} = 30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x}.
\]

Hence,
\[
R = \text{rank } \mathcal{R} = \text{rank } G^{(3)} - \text{rank } G^{(2)} = \text{rank } G^{(2)} - \text{rank } G^{(1)} = 2.
\]

Step 2: Construct the form of the recursion operator.

(i) Determine the pieces of operator \( \mathcal{R}_0 \)

List all permutations of type \( D^j u^k, u_x^l D^j \), etc. of rank \( R \)
\((j, k, l \) nonnegative integers).
\[
\mathcal{L} = \{D^2, u\}.
\]

(ii) Determine the pieces of operator \( \mathcal{R}_1 \)

Combine the symmetries \( G^{(j)} \) with \( D^{-1} \) and \( \rho_{(k)}'(u) \), so that every
term in
\[
\mathcal{R}_1 = \sum_j \sum_k G^{(j)} D^{-1} \rho_{(k)}'(u)
\]
has rank \( R \).

The indices \( j \) and \( k \) are taken so that
\[
\text{rank } (G^{(j)}) + \text{rank } (\rho_{(k)}'(u)) - 1 = R.
\]

List such terms:
\[
\mathcal{M} = \{u_x D^{-1}\}.
\]
(iii) **Build the operator** $\mathcal{R}$

Linearly combine the term in

$$\mathcal{R} = \mathcal{L} \cup \mathcal{M} = \{D^2, uI, u_xD^{-1}\}.$$  

to get

$$\mathcal{R} = c_1 D^2 + c_2 uI + c_3 u_xD^{-1}.$$  

**Step 3: Determine the unknown coefficients.**

Substitute in the determining equation, alternatively, require that

$$\mathcal{R}G^{(k)} = G^{(k+1)}, \quad k = 1, 2, 3, ...$$

Solve the linear system:

$$\mathcal{S} = \{c_1-1 = 0, 18c_1+c_3-20 = 0, 6c_1+c_2-10 = 0, 2c_2+c_3-10 = 0\},$$

Solution: $c_1 = 1, c_2 = 4, \text{ and } c_3 = 2$. So,

$$\mathcal{R} = D^2 + 4 uI + 2 u_xD^{-1}.$$  

**Example.**

Sawada-Kotera (SK) equation:

$$u_t = 5u^2u_x + 5u_xu_{2x} + 5uu_{3x} + u_{5x}.$$  

Recursion operator:

$$\mathcal{R} = D^6 + 3uD^4 - 3uD^3 + 11D^2uD^2 - 10D^3uD + 5D^4uI$$

$$+ 12u^2D^2 - 19uD^2D + 8uD^2uI + 8DuDuI + 4u^3I$$

$$+ u_xD^{-1}(u^2 - 2u_xD) + G^{(2)}D^{-1},$$

with $G^{(2)} = 5u^2u_x + 5u_xu_{2x} + 5uu_{3x} + u_{5x}$. 

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Matrix Case

Recursion operator (matrix) splits naturally in \( \mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1 \).

The rank of the entries of the recursion operator (matrix):

\[
\text{rank}(\mathcal{R}_{ij}) = \text{rank}(G_i^{(k+1)}) - \text{rank}(G_j^{(k)}),
\]

where \( G_i^{(k)} \) is \( ith \) component of \( kth \) symmetry.

Entries of \( \mathcal{R}_0 \) are linear combinations of \((u, u_x, u_{2x}, \ldots)\) and \((I, D_x, D_x^2, \ldots)\) of rank \( R \).

Matrix \( \mathcal{R}_1 \) is of the form

\[
\sum_j \sum_k G^{(j)} D_x^{-1} \otimes \rho'(k)
\]

where \( \otimes \) denotes the matrix outer product, and \( \rho'(k) \) is the Fréchet derivative of \( \rho(k) \).

Example.

The vector nonlinear Schrödinger system:

\[
u_t + [\nu(u^2 + v^2) + \beta u + \gamma v - v_x]_x = 0,
\]

\[
u_t + [v(u^2 + v^2) + \theta u + \delta v + u_x]_x = 0.
\]

Recursion operator:

\[
\mathcal{R} = \begin{pmatrix}
(\beta - \delta + 2u^2)I + 2u_x D^{-1} u I & (\theta + 2uv)I - D + 2u_x D^{-1} v I \\
(\theta + 2uv)I + D + 2v_x D^{-1} u I & 2v^2I + 2v_x D^{-1} v I
\end{pmatrix}.
\]
Analogy PDEs and DDEs

Conservation laws for PDEs

\[ D_t \rho + D_x J = 0 \]

density \( \rho \), flux \( J \).

Compute \( E = D_t \rho \).

Use PDE to replace all \( t \)-derivatives: \( u_t, u_{tx}, u_{txx}, \) etc.

To avoid integration by parts, apply the continuous Euler operator

\[ \mathcal{L}_u = \sum_{i=0}^{m} (-1)^i D_x^i \frac{\partial}{\partial u_{ix}} \]

\[ = \frac{\partial}{\partial u} - D_x (\frac{\partial}{\partial u_x}) + D_x^2 (\frac{\partial}{\partial u_{2x}}) + \cdots + (-1)^m D_x^m (\frac{\partial}{\partial u_{mx}}). \]

to \( E \) of order \( m \).

\( D_x \) is the differential operator.

If \( \mathcal{L}_u(E) = 0 \), then \( E \) is a total \( x \)-derivative \( (-J_x) \).

If \( \mathcal{L}_u(E) \neq 0 \), the nonzero terms must vanish identically.

\( E \) must be in the kernel of \( \mathcal{L}_u \) operator, or equivalently, \( E \) must be in the image of \( D_x \) operator.
Conservation laws for DDEs

\[ \dot{\rho}_n + J_{n+1} - J_n = 0 \]

density \( \rho_n \), flux \( J_n \).

Compute \( E = \dot{\rho}_n \).

Use the DDE to remove all \( t \)-derivatives, \( \dot{u}_n \dot{u}_{n \pm 1}, \dot{u}_{n \pm 2}, \) etc.

To avoid pattern matching, apply the discrete Euler operator

\[
\mathcal{L}_{un} = \sum_{i=-q}^{p} D^{-i} \frac{\partial}{\partial u_{n+i}} \\
= \frac{\partial}{\partial u_n} + D\left(\frac{\partial}{\partial u_{n-1}}\right) + D^2\left(\frac{\partial}{\partial u_{n-2}}\right) + \cdots + D^q\left(\frac{\partial}{\partial u_{n-q}}\right) \\
+ D^{-1}\left(\frac{\partial}{\partial u_{n+1}}\right) + D^{-2}\left(\frac{\partial}{\partial u_{n+2}}\right) + \cdots + D^{-p}\left(\frac{\partial}{\partial u_{n+p}}\right)
\]

to \( E \) with maximal shifts \( n - q, n + p \).

\( D \) is the up-shift operator, \( D^{-1} \) the down-shift operator.

Applied to a monomial \( m \)

\[ D^{-1}m = m|_{n\rightarrow n-1} \quad \text{and} \quad Dm = m|_{n\rightarrow n+1}. \]

Note: \( D \) (up-shift operator) corresponds the differential operator \( D_x \) due to the forward difference

\[ \frac{\partial J}{\partial x} \rightarrow \frac{J_{n+1} - J_n}{\Delta x} \quad (\Delta x = 1) \]

If \( \mathcal{L}_{un}(E) = 0 \), then \( E \) matches \( -(J_{n+1} - J_n) \).

If \( \mathcal{L}_{un}(E) \neq 0 \), the nonzero terms must vanish identically.
Part III  Algorithms for DDEs (lattices)

• Extra Tool: Equivalence Criterion

$D^{-1}$ and $D$ are the down-shift and up-shift operators.

For a monomial $m$:

$$D^{-1}m = m|_{n 	o n-1}, \quad \text{and} \quad Dm = m|_{n 	o n+1}.$$  

**Example**

$$D^{-1}u_{n+2}v_n = u_{n+1}v_{n-1}, \quad Du_{n-2}v_{n-1} = u_{n-1}v_n.$$  

Compositions of $D^{-1}$ and $D$ define an equivalence relation. All shifted monomials are equivalent.

**Example**

$$u_{n-1}v_{n+1} \equiv u_{n+2}v_{n+4} \equiv u_{n-3}v_{n-1}.$$  

**Equivalence criterion:**

Two monomials $m_1$ and $m_2$ are equivalent, $m_1 \equiv m_2$, if

$$m_1 = m_2 + [M_n - M_{n+1}]$$

for some polynomial $M_n$.

For example, $u_{n-2}u_n \equiv u_{n-1}u_{n+1}$ since

$$u_{n-2}u_n = u_{n-1}u_{n+1} + [u_{n-2}u_n - u_{n-1}u_{n+1}] = u_{n-1}u_{n+1} + [M_n - M_{n+1}].$$

**Main representative** of an equivalence class is the monomial with label $n$ on $u$ (or $v$).

For example, $u_n u_{n+2}$ is the main representative of the class with elements $u_{n-1}u_{n+1}, u_{n+1}u_{n+3},$ etc.

Use lexicographical ordering to resolve conflicts.

For example, $u_n v_{n+2}$ (not $u_{n-2}v_n$) is the main representative of the class with elements $u_{n-3}v_{n-1}, u_{n+2}v_{n+4},$ etc.
• **Algorithm for Conserved Densities of DDEs.**

Three-step algorithm to find conserved densities:
(i) Determine the weights.
(ii) Construct the form of density.
(iii) Determine the coefficients.

**Example:** Density of rank 3 of the Toda lattice,

\[ \dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}). \]

**Step 1: Compute the weights.**

Require uniformity in rank for each equation:

\[ w(u_n) + w\left(\frac{d}{dt}\right) = w(v_{n-1}) = w(v_n), \]
\[ w(v_n) + w\left(\frac{d}{dt}\right) = w(v_n) + w(u_n) = w(v_n) + w(u_{n+1}) \]

Weights are shift invariant. Set \( w\left(\frac{d}{dt}\right) = 1 \) and solve the linear system: \( w(u_n) = w(u_{n+1}) = 1 \) and \( w(v_n) = w(v_{n-1}) = 2 \).

**Step 2: Construct the form of the density.**

List all monomials in \( u_n \) and \( v_n \) of rank 3 or less:

\[ G = \{ u_n^3, u_n^2, u_n v_n, u_n, v_n \}. \]

For each monomial in \( G \), introduce enough \( t \)-derivatives to obtain weight 3. Use the DDE to remove \( \dot{u}_n \) and \( \dot{v}_n \):

\[ \frac{d^0}{dt^0}(u_n^3) = u_n^3, \quad \frac{d^0}{dt^0}(u_n v_n) = u_n v_n, \]
\[ \frac{d}{dt}(u_n^2) = 2u_n v_{n-1} - 2u_n v_n, \]

---

1In general algorithm shifts are also needed: \( u_n^3, u_n u_{n+1} u_{n-1}, u_n^2 u_{n+1}, \) etc.
\[
\frac{d}{dt}(v_n) = u_n v_n - u_{n+1} v_n,
\]
\[
\frac{d^2}{dt^2}(u_n) = u_{n-1} v_{n-1} - u_n v_{n-1} - u_n v_n + u_{n+1} v_n.
\]

Gather the resulting terms in a set

\[
\mathcal{H} = \{ u_n^3, u_n v_{n-1}, u_n v_n, u_{n-1} v_{n-1}, u_{n+1} v_n \}.
\]

Replace members in the same equivalence class by their main representatives.

For example, \( u_n v_{n-1} \equiv u_{n+1} v_n \) are replaced by \( u_n v_{n-1} \).

Linearly combine the monomials in

\[
\mathcal{I} = \{ u_n^3, u_n v_{n-1}, u_n v_n \}
\]

to obtain

\[
\rho_n = c_1 u_n^3 + c_2 u_n v_{n-1} + c_3 u_n v_n.
\]

**Step 3: Determine the coefficients.**

Require that \( \dot{\rho}_n + J_{n+1} - J_n = 0 \) holds.

Compute \( \dot{\rho}_n \). Use the DDE to remove \( \dot{u}_n \) and \( \dot{v}_n \). Thus,

\[
E = \dot{\rho}_n = (3c_1 - c_2) u_n^2 v_{n-1} + (c_3 - 3c_1) u_n^2 v_n + (c_3 - c_2) v_{n-1} v_n
\]

\[
+ c_2 u_{n-1} u_n v_{n-1} + c_2 v_{n-1}^2 - c_3 u_n u_{n+1} v_n - c_3 v_n^2.
\]

Apply the discrete Euler operator

\[
\mathcal{L}_{u_n} = \sum_{i=-q}^{p} D^{-i} \frac{\partial}{\partial u_{n+i}}
\]

\[
= \frac{\partial}{\partial u_n} + D\left( \frac{\partial}{\partial u_{n-1}} \right) + D^2\left( \frac{\partial}{\partial u_{n-2}} \right) + \cdots + D^q\left( \frac{\partial}{\partial u_{n-q}} \right)
\]

\[
+ D^{-1}\left( \frac{\partial}{\partial u_{n+1}} \right) + D^{-2}\left( \frac{\partial}{\partial u_{n+2}} \right) + \cdots + D^{-p}\left( \frac{\partial}{\partial u_{n+p}} \right)
\]
to $E$ with maximal shifts $n - 1$, $n + 1$.

$$\mathcal{L}_{u_n}(E) = (3c_1 - c_2)u_n^2v_{n-1} + (c_3 - 3c_1)u_n^2v_n + (c_3 - c_2)v_nv_{n+1} + (c_2 - c_3)v_n^2$$

Solve the linear system

$$\mathcal{S} = \{3c_1 - c_2 = 0, c_3 - 3c_1 = 0, c_2 - c_3 = 0\}.$$  

The solution is $3c_1 = c_2 = c_3$. Choose $c_1 = \frac{1}{3}$, and $c_2 = c_3 = 1$.

Use the equivalence criterion and main representatives to rearrange $E$ to match the pattern $[J_n] - [J_{n+1}]$.

$$E = [u_{n-1}u_nv_{n-1} + v_{n-1}^2] - [u_nu_{n+1}v_n + v_n^2].$$

Hence,

$$\rho_n = \frac{1}{3} u_n^3 + u_n(v_{n-1} + v_n), \quad J_n = u_{n-1}u_nv_{n-1} + v_{n-1}^2.$$  

Analogously, conserved densities of rank $\leq 5$:

$$\rho_n^{(1)} = u_n \quad \rho_n^{(2)} = \frac{1}{2}u_n^2 + v_n$$

$$\rho_n^{(3)} = \frac{1}{3}u_n^3 + u_n(v_{n-1} + v_n)$$

$$\rho_n^{(4)} = \frac{1}{4}u_n^4 + u_n^2(v_{n-1} + v_n) + u_nu_{n+1}v_n + \frac{1}{2}v_n^2 + v_nv_{n+1}$$

$$\rho_n^{(5)} = \frac{1}{5}u_n^5 + u_n^3(v_{n-1} + v_n) + u_nu_{n+1}v_n(u_n + u_{n+1})$$

$$\quad + u_nv_{n-1}(v_{n-2} + v_{n-1} + v_n) + u_nv_n(v_{n-1} + v_n + v_{n+1}).$$
• Algorithm for Generalized Symmetries of DDEs.

Consider the Toda system
\[ \dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}). \]

with
\[ w(u_n) = 1 \quad \text{and} \quad w(v_n) = 2. \]
Compute the form of the symmetry of ranks \((3, 4)\), i.e. the first component of the symmetry has rank 3, the second rank 4.

**Step 1: Construct the form of the symmetry.**

List all monomials in \(u_n\) and \(v_n\) of rank 3 or less:
\[ \mathcal{L}_1 = \{u_n^3, u_n^2, u_nv_n, u_n, v_n\}, \]
and of rank 4 or less:
\[ \mathcal{L}_2 = \{u_n^4, u_n^3, u_n^2v_n, u_n^2, u_nv_n, u_n, v_n^2, v_n\}. \]

For each monomial in \(\mathcal{L}_1\) and \(\mathcal{L}_2\), introduce enough \(t\)-derivatives, so that each term exactly has rank 3 and 4, respectively.

Using the DDEs, for the monomials in \(\mathcal{L}_1\):
\[
\begin{align*}
\frac{d^0}{dt^0}(u_n^3) &= u_n^3, & \frac{d^0}{dt^0}(u_nv_n) &= u_nv_n, \\
\frac{d}{dt}(u_n^2) &= 2u_n\dot{u}_n = 2u_nv_{n-1} - 2u_nv_n, \\
\frac{d}{dt}(v_n) &= \dot{v}_n = u_nv_n - u_{n+1}v_n, \\
\frac{d^2}{dt^2}(u_n) &= \frac{d}{dt}(\dot{u}_n) = \frac{d}{dt}(v_{n-1} - v_n) \\
&= u_{n-1}v_{n-1} - u_nv_{n-1} - u_nv_n + u_{n+1}v_n.
\end{align*}
\]
Gather the resulting terms:
\[ \mathcal{R}_1 = \{ u_n^3, u_{n-1}v_{n-1}, u_nv_{n-1}, u_nv_n, u_{n+1}v_n \}. \]
\[ \mathcal{R}_2 = \{ u_n^4, u_{n-1}v_{n-1}, u_{n-1}u_nv_{n-1}, u_n^2v_{n-1}, v_{n-2}v_{n-1}, v_{n-1}^2, u_n^2v_n, u_nu_{n+1}v_n, u_{n+1}v_n, v_{n-1}v_n, v_n^2, v_nv_{n+1} \}. \]

Linearly combine the monomials in $\mathcal{R}_1$ and $\mathcal{R}_2$
\[ G_1 = c_1 u_n^3 + c_2 u_{n-1}v_{n-1} + c_3 u_nv_{n-1} + c_4 u_nv_n + c_5 u_{n+1}v_n, \]
\[ G_2 = c_6 u_n^4 + c_7 u_{n-1}^2v_{n-1} + c_8 u_{n-1}u_nv_{n-1} + c_9 u_n^2v_{n-1} + c_{10} v_{n-2}v_{n-1} + c_{11} v_{n-1}^2 + c_{12} u_n^2v_n + c_{13} u_nu_{n+1}v_n + c_{14} u_{n+1}^2v_n + c_{15} v_{n-1}v_n + c_{16} v_n^2 + c_{17} v_nv_{n+1}. \]

**Step 2: Determine the unknown coefficients.**

Require that the symmetry condition $D_t G = F'(u_n)[G]$ holds.

Solution:
\[ c_1 = c_6 = c_7 = c_8 = c_9 = c_{10} = c_{11} = c_{13} = c_{16} = 0, \]
\[ -c_2 = -c_3 = c_4 = c_5 = -c_{12} = c_{14} = -c_{15} = c_{17}. \]

Therefore, with $c_{17} = 1$, the symmetry of rank $(3, 4)$ is:
\[ G_1 = u_nv_n - u_{n-1}v_{n-1} + u_{n+1}v_n - u_nv_{n-1}, \]
\[ G_2 = u_{n+1}^2v_n - u_n^2v_n + v_nv_{n+1} - v_{n-1}v_n. \]

Analogously, the symmetry of rank $(4, 5)$ reads
\[ G_1 = u_n^2v_n + u_nu_{n+1}v_n + u_{n+1}^2v_n + v_n^2 + v_nv_{n+1} - u_{n-1}^2v_{n-1} - u_{n-1}u_nv_{n-1} - u_n^2v_{n-1} - v_{n-2}v_{n-1} - v_{n-1}^2, \]
\[ G_2 = u_{n+1}v_n^2 + 2u_{n+1}v_nv_{n+1} + u_{n+2}v_nv_{n+1} - u_n^3v_n + u_{n+1}^3v_n - u_{n-1}v_{n-1}v_n - 2u_nv_{n-1}v_n - u_n^2v_n. \]
• Algorithm for Recursion Operators of DDEs.

Scalar Case

Similar to the continuous case.

$D^{-1}$ and $D$ are down and up-shift operators.

$I$ is the identity operator.

$D - I$ is the discretized version of $D_x$ (PDE case).

$(D - I)^{-1}$ corresponding to the integral operator $D_x^{-1}$ (PDE case).

Recursion operator splits in $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$.

$\mathcal{R}_0$ has combinations of $D^{-1}, D,$ and $u_{n\pm p}$.

Matrix $\mathcal{R}_1$ is of the form

$$\mathcal{R}_1 = \sum_j \sum_k G^{(j)}(D - I)^{-1} \rho'_(k)$$

Example

The Kac-van Moerbeke equation

$$\dot{u}_n = u_n(u_{n+1} - u_{n-1}).$$

Recursion operator:

$$\mathcal{R} = u_n D + u_n D^{-1} + (u_n + u_{n+1})I + u_n(u_{n+1} - u_{n-1})(D - I)^{-1} \frac{1}{u_n} I$$

$$= u_n(I + D)(u_n D - D^{-1} u_n)(D - I)^{-1} \frac{1}{u_n} I$$
Matrix Case

Recursion operator (matrix) splits naturally in $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$. Entries of matrix $\mathcal{R}_0$ are linear combinations of $(u_n, u_{n \pm 1}, u_{n \pm 2}, ...)$ and $(I, D, D^{-1}, ...)$ of rank $R$.

Matrix $\mathcal{R}_1$ is of the form

$$\sum_j \sum_k G^{(j)}(D - I)^{-1} \otimes \rho'_k$$

where $\otimes$ denotes the matrix outer product, and $\rho'_k$ is the Fréchet derivative of $\rho_k$.

Example.

The Toda lattice

$$\dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n (u_n - u_{n+1}).$$

Recursion operator:

$$\mathcal{R} = \begin{pmatrix} -u_n I & -D^{-1} - I + (v_{n-1} - v_n)(D - I)^{-1} \frac{1}{v_n} I \\ -v I - vD & u_{n+1} I + v_n (u_n - u_{n+1})(D - I)^{-1} \frac{1}{v_n} I \end{pmatrix}$$
• Example: The Ablowitz-Ladik DDE.

Consider the Ablowitz and Ladik discretization,

$$i \dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + \kappa u_n^* u_n (u_{n+1} + u_{n-1}),$$

of the NLS equation,

$$iu_t + u_{xx} + \kappa u^2 u^* = 0$$

$u_n^*$ is the complex conjugate of $u_n$. Treat $u_n$ and $v_n = u_n^*$ as independent variables and add the complex conjugate equation. Set $\kappa = 1$ (scaling) and absorb $i$ in the scale on $t$:

$$\dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + u_n v_n (u_{n+1} + u_{n-1}),$$

$$\dot{v}_n = -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}).$$

Since $v_n = u_n^*$, $w(v_n) = w(u_n)$.

No uniformity in rank! Introduce an auxiliary parameter $\alpha$ with weight.

$$\dot{u}_n = \alpha (u_{n+1} - 2u_n + u_{n-1}) + u_n v_n (u_{n+1} + u_{n-1}),$$

$$\dot{v}_n = -\alpha (v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}).$$

Uniformity in rank leads to

$$w(u_n) + w\left(\frac{d}{dt}\right) = w(\alpha) + w(u_n) = 2w(u_n) + w(v_n),$$

$$w(v_n) + w\left(\frac{d}{dt}\right) = w(\alpha) + w(v_n) = 2w(v_n) + w(u_n).$$

For $w\left(\frac{d}{dt}\right) = 1$,

$$w(u_n) + w(v_n) = w(\alpha) = 1.$$ 

So, one solution is

$$w(u_n) = w(v_n) = \frac{1}{2}, \quad w(\alpha) = 1.$$
Alternatively, for \( w\left( \frac{d}{dt} \right) = 0 \),
\[
  w(u_n) + w(v_n) = 0, \quad w(\alpha) = 0.
\]

The second scale helps eliminate terms in candidate density \( \rho \).

Conserved densities (for \( \alpha = 1 \), in original variables):
\[
  \rho^{(1)}_n = u_n u_{n-1}^*
\]
\[
  \rho^{(2)}_n = u_n u_{n+1}^*
\]
\[
  \rho^{(3)}_n = \frac{1}{2} u_n^2 u_{n-1}^2 + u_n u_{n+1} u_{n-1}^* v_n + u_n u_{n-2}^*
\]
\[
  \rho^{(4)}_n = \frac{1}{2} u_n^2 u_{n+1}^2 + u_n u_{n+1} u_{n+2}^* + u_n u_{n+2}^*
\]
\[
  \rho^{(5)}_n = \frac{1}{3} u_n^3 u_{n-1}^3 + u_n u_{n+1} u_{n-1}^* u_n^* (u_n u_{n-1}^* + u_{n+1}^* + u_{n+2}^* u_{n+1}^*)
  + u_n u_{n-1}^* (u_n u_{n-2}^* + u_{n+1} u_{n-1}^*) + u_n u_n^* (u_{n+1} u_{n-2}^* + u_{n+2}^* u_{n-1}^*) + u_n u_{n-3}^*
\]
\[
  \rho^{(6)}_n = \frac{1}{3} u_n^3 u_{n+1}^3 + u_n u_{n+1} u_{n+2}^* (u_n u_{n+1}^* + u_{n+1} u_{n+2}^* + u_{n+2} u_{n+3}^*)
  + u_n u_{n+2}^* (u_n u_{n+1}^* + u_{n+1} u_{n+2}^*) + u_n u_{n+3}^* (u_{n+1} u_{n+1}^* + u_{n+2} u_{n+2}^*) + u_n u_{n+3}^*
\]

The Ablowitz-Ladik lattice has infinitely many conserved densities.

Density we missed
\[
  \rho^{(0)}_n = \ln(1 + u_n u_n^*).
\]

We cannot find the Hamiltonian (constant of motion):
\[
  H = -i \sum [u_n^*(u_{n-1} + u_{n+1}) - 2 \ln(1 + u_n u_n^*)],
\]
since it has a logarithmic term.
• Application: Discretization of combined KdV-mKdV equation.

Consider the integrable discretization
\[ \dot{u}_n = -(1 + \alpha h^2 u_n + \beta h^2 u_n^2) \left\{ \frac{1}{h^3} \left( \frac{1}{2} u_{n+2} - u_{n+1} + u_{n-1} - \frac{1}{2} u_{n-2} \right) 
+ \frac{\alpha}{2h} \left[ u_{n+1}^2 - u_{n-1}^2 + u_n(u_{n+1} - u_{n-1}) + u_{n+1}u_{n+2} - u_{n-1}u_{n-2} \right] 
+ \frac{\beta}{2h} \left[ u_{n+1}^2(u_{n+2} + u_n) - u_{n-1}^2(u_{n-2} + u_n) \right] \right\} \]

of a combined KdV-mKdV equation
\[ u_t + 6\alpha uu_x + 6\beta u_x^2 u + u_{xxx} = 0. \]
Discretizations the KdV and mKdV equations are special cases.
Set \( h = 1 \) (scaling). No uniformity in rank!

Introduce auxiliary parameters \( \gamma \) and \( \delta \) with weights.

\[ \dot{u}_n = -(\gamma + \alpha u_n + \beta u_n^2) \left\{ \delta \left( \frac{1}{2} u_{n+2} - u_{n+1} + u_{n-1} - \frac{1}{2} u_{n-2} \right) 
+ \frac{\alpha}{2} \left[ u_{n+1}^2 - u_{n-1}^2 + u_n(u_{n+1} - u_{n-1}) + u_{n+1}u_{n+2} - u_{n-1}u_{n-2} \right] 
+ \frac{\beta}{2} \left[ u_{n+1}^2(u_{n+2} + u_n) - u_{n-1}^2(u_{n-2} + u_n) \right] \right\}, \]

Uniformity in rank requires
\[ w(\gamma) = w(\delta) = 2w(u_n), \quad w(\alpha) = w(u_n), \quad w(\beta) = 0. \]

Then,
\[ w(u_n) + 1 = 5w(u_n), \]

Hence,
\[ w(u_n) = w(\alpha) = \frac{1}{4}, \quad w(\gamma) = w(\delta) = \frac{1}{2}, \quad w(\beta) = 0, \]
Conserved densities: Special cases

For the KdV case ($\beta = 0$):

$$\dot{u}_n = -\left(\gamma + \alpha h^2 u_n\right) \left\{ \frac{\delta}{h^3} \left(\frac{1}{2} u_{n+2} - u_{n+1} + u_{n-1} - \frac{1}{2} u_{n-2}\right) + \frac{\alpha}{2h} \left[u_{n+1}^2 - u_{n-1}^2 + u_n(u_{n+1} - u_{n-1}) + u_{n+1}u_{n+2} - u_{n-1}u_{n-2}\right] \right\}$$

with $\gamma = \delta = 1$ is a completely integrable discretization of the KdV equation

$$u_t + 6\alpha uu_x + u_{xxx} = 0.$$ 

Now,

$$w(\gamma) = w(\delta) = w(u_n), \quad w(\alpha) = 0.$$ 

Then,

$$w(u_n) + 1 = 3w(u_n).$$ 

So,

$$w(u_n) = w(\gamma) = w(\delta) = \frac{1}{2}, \quad w(\alpha) = 0.$$ 

From rank $\frac{3}{2}$ and $\frac{5}{2}$ (after splitting):

$$\rho_n^{(1)} = u_n,$$

$$\rho_n^{(2)} = u_n\left(\frac{1}{2}u_n + u_{n+1}\right),$$

$$\rho_n^{(3)} = u_n\left(\frac{1}{3}u_n^2 + u_nu_{n+1} + u_{n+1}^2 + \frac{1}{\alpha}u_{n+2} + u_{n+1}u_{n+2}\right),$$

$$\rho_n^{(4)} = u_n\left(\frac{1}{4}u_n^3 + u_n^2u_{n+1} + \frac{3}{2}u_nu_{n+1}^2 + u_{n+1}^3 + \cdots + u_{n+1}u_{n+2}u_{n+3}\right),$$

$$\rho_n^{(5)} = u_n\left(\frac{1}{5}\alpha u_n^4 - \frac{1}{2}u_n^3 - 2u_n^2u_{n+1} + \cdots + \alpha u_{n+1}u_{n+2}u_{n+3}u_{n+4}\right)$$
For the mKdV case ($\alpha = 0$):
\[
\dot{u}_n = -(\gamma + \beta h^2 u_n^2) \left\{ \frac{\delta}{h^3}(\frac{1}{2}u_{n+2} - u_{n+1} + u_{n-1} - \frac{1}{2}u_{n-2}) + \frac{\beta}{2h}[u_{n+1}(u_{n+2} + u_n) - u_{n-1}(u_{n-2} + u_n)] \right\}
\]

with $\gamma = \delta = 1$ is a completely integrable discretization of the modified KdV equation
\[
u_t + 6\beta u^2 v_x + v_{xxx} = 0.
\]
Now,
\[
w(\gamma) = w(\delta) = 2w(u_n), \quad w(\beta) = 0.
\]
Then,
\[
w(u_n) + 1 = 5w(u_n).
\]
So,
\[
w(u_n) = \frac{1}{4}, \quad w(\gamma) = w(\delta) = \frac{1}{2}, \quad w(\beta) = 0.
\]
From rank $\frac{3}{2}$ and $\frac{5}{2}$ (after splitting):
\[
\rho_n^{(1)} = u_n u_{n+1},
\]
\[
\rho_n^{(2)} = u_n(\frac{1}{2}u_n u_{n+1}^2 + \frac{1}{\beta} u_{n+2} + u_{n+1}u_{n+2})
\]
\[
\rho_n^{(3)} = u_n(\frac{1}{3}u_n^2 u_{n+1}^3 + \frac{1}{3}u_n u_{n+1} u_{n+2} + \cdots + u_{n+1}^2 u_{n+2} u_{n+3})
\]
\[
\rho_n^{(4)} = u_n(\frac{1}{4}\beta u_n^3 u_{n+1}^4 + u_n^2 u_{n+1} u_{n+2} + \cdots + \beta u_{n+1}^2 u_{n+2}^2 u_{n+3} u_{n+4})
\]
Part IV  Software, Future Work, Publications

• **Scope and Limitations of Algorithms.**
  
  – Systems of PDEs and DDEs must be polynomial in dependent variables. No *explicitly* dependencies on the independent variables. (Transcendental nonlinearities in progress).

  – Currently, one space variable (continuous or discretized).

  – Program only computes polynomial conservation laws and generalized symmetries (no recursion operators yet).

  – Program computes conservation laws and symmetries that explicitly depend on the independent variables, if the highest degree is specified.

  – No limit on the number of equations in the system. In practice: time and memory constraints.

  – Input systems may have (nonzero) parameters. Program computes the compatibility conditions for parameters such that conservation laws and symmetries (of a given rank) exist.

  – Systems can also have parameters with (unknown) weight. This allows one to test evolution and lattice equations of non-uniform rank.

  – For systems where one or more of the weights is free, the program prompts the user for info.

  – Fractional weights and ranks are permitted.

  – Complex dependent variables are allowed.

  – PDEs and lattice equations must be of first-order in $t$. 


• **Conclusions and Future Research**

  – Implement the recursion operator algorithm for PDEs and DDEs.

  – Generalization to (3+1)-dimensional case
    (e.g. Kadomtsev-Petviashvili equation).

  – Computation of first integrals of ODEs, constants of motion for dynamical systems (e.g. Lorenz, Hénon-Heiles, Rossler systems).

  – Improve software, compare with other packages.

  – Add tools for parameter analysis (Gröbner basis, Ritt-Wu or characteristic sets algorithms).

  – Generalization towards broader classes of equations (e.g. $u_{xt}$).

  – Exploit other symmetries in the hope to find conserved densities of non-polynomial form

  – Application: test model DDEs for integrability.
    (study the integrable discretization of KdV-mKdV equation).
• **Implementation in Mathematica – Software**


  – Software: available via FTP, ftp site *mines.edu* in

    pub/papers/math_cs_dept/software/condens

    pub/papers/math_cs_dept/software/diffdens

  or via the Internet

    URL: http://www.mines.edu/fs_home/whereman/
• Publications


Application

A Class of Fifth-Order Evolution Equations

\[ u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0 \]

where \( \alpha, \beta, \gamma \) are nonzero parameters.

\[ u \sim D_x^2. \]

Special cases:

\[
\begin{align*}
\alpha &= 30 \quad \beta = 20 \quad \gamma = 10 \quad \text{Lax.} \\
\alpha &= 5 \quad \beta = 5 \quad \gamma = 5 \quad \text{Sawada – Kotera.} \\
\alpha &= 20 \quad \beta = 25 \quad \gamma = 10 \quad \text{Kaup–Kopershmidt.} \\
\alpha &= 2 \quad \beta = 6 \quad \gamma = 3 \quad \text{Ito.}
\end{align*}
\]

What are the conditions for the parameters \( \alpha, \beta \) and \( \gamma \) so that the equation admits a density of fixed rank?

- **Rank 2:**
  No condition
  \[ \rho = u. \]

- **Rank 4:**
  Condition: \( \beta = 2\gamma \) (Lax and Ito cases)
  \[ \rho = u^2. \]
• **Rank 6:**

  Condition:

  \[ 10\alpha = -2\beta^2 + 7\beta\gamma - 3\gamma^2 \]

  (Lax, SK, and KK cases)

  \[ \rho = u^3 + \frac{15}{(-2\beta + \gamma)} u_x^2. \]

• **Rank 8:**

  1). \( \beta = 2\gamma \)  (Lax and Ito cases)

  \[ \rho = u^4 - \frac{6\gamma}{\alpha} uu_x^2 + \frac{6}{\alpha} u_{2x}^2. \]

  2). \( \alpha = \frac{-2\beta^2 - 7\beta\gamma - 4\gamma^2}{45} \)  (SK, KK and Ito cases)

  \[ \rho = u^4 - \frac{135}{2\beta + \gamma} uu_x^2 + \frac{675}{(2\beta + \gamma)^2} u_{2x}^2. \]

• **Rank 10:**

  Condition:

  \( \beta = 2\gamma \)

  and

  \( 10\alpha = 3\gamma^2 \)

  (Lax case)

  \[ \rho = u^5 - \frac{50}{\gamma^3} u^2 u_x^2 + \frac{100}{\gamma^2} uu_{2x}^2 - \frac{500}{7\gamma^3} u_{3x}^2. \]
What are the necessary conditions for the parameters $\alpha, \beta$ and $\gamma$ so that the equation admits $\infty$ many polynomial conservation laws?

- If $\alpha = \frac{3}{10} \gamma^2$ and $\beta = 2 \gamma$ then there is a sequence (without gaps!) of conserved densities (Lax case).

- If $\alpha = \frac{1}{5} \gamma^2$ and $\beta = \gamma$ then there is a sequence (with gaps!) of conserved densities (SK case).

- If $\alpha = \frac{1}{5} \gamma^2$ and $\beta = \frac{5}{2} \gamma$ then there is a sequence (with gaps!) of conserved densities (KK case).

- If

  $$\alpha = -\frac{2 \beta^2 - 7 \beta \gamma + 4 \gamma^2}{45}$$

  or

  $$\beta = 2 \gamma$$

  then there is a conserved density of rank 8.

Combine both conditions: $\alpha = \frac{2 \gamma^2}{9}$ and $\beta = 2 \gamma$ (Ito case).

SUMMARY: see tables (notice the gaps!)
<table>
<thead>
<tr>
<th>Density</th>
<th>Sawada-Kotera equation</th>
<th>Lax equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho(1)$</td>
<td>$u$</td>
<td>$u$</td>
</tr>
<tr>
<td>$\rho(2)$</td>
<td>$----$</td>
<td>$\frac{1}{2} u^2$</td>
</tr>
<tr>
<td>$\rho(3)$</td>
<td>$\frac{1}{3} u^3 - u_x^2$</td>
<td>$\frac{1}{3} u^3 - \frac{1}{6} u_x^2$</td>
</tr>
<tr>
<td>$\rho(4)$</td>
<td>$\frac{1}{4} u^4 - \frac{9}{4} u u_x^2 + \frac{3}{4} u_{2x}^2$</td>
<td>$\frac{1}{4} u^4 - \frac{1}{2} u u_x^2 + \frac{1}{20} u_{2x}^2$</td>
</tr>
<tr>
<td>$\rho(5)$</td>
<td>$----$</td>
<td>$\frac{1}{5} u^5 - u^2 u_x^2 + \frac{1}{5} u u_{2x}^2 - \frac{1}{10} u_{3x}^2$</td>
</tr>
<tr>
<td>$\rho(6)$</td>
<td>$\frac{1}{6} u^6 - \frac{25}{4} u^3 u_x^2 - \frac{17}{8} u_x^4 + 6 u^2 u_{2x}^2 + 2 u_{2x}^3 - \frac{21}{8} u u_{3x}^2 + \frac{3}{8} u_{4x}^2$</td>
<td>$\frac{1}{6} u^6 - \frac{5}{4} u^3 u_x^2 - \frac{5}{36} u_x^4 + \frac{1}{2} u u_{2x}^2 + \frac{5}{63} u_{2x}^3 - \frac{1}{14} u u_{3x}^2 + \frac{1}{252} u_{4x}^2$</td>
</tr>
<tr>
<td>$\rho(7)$</td>
<td>$\frac{1}{7} u^7 - 9 u^4 u_x^2 - \frac{54}{5} u u_x^4 + \frac{57}{5} u^3 u_{2x}^2 + \frac{648}{35} u x^2 u_{2x}^2 + \frac{489}{35} u u_{2x}^3 - \frac{261}{35} u^2 u_{3x}^2 - \frac{288}{35} u_{2x}^3 u_{3x}^2 + \frac{81}{35} u u_{4x}^2 - \frac{9}{35} u_{5x}^2$</td>
<td>$\frac{1}{7} u^7 - \frac{5}{2} u^4 u_x^2 - \frac{5}{6} u u_x^4 + u^3 u_{2x}^2 + \frac{1}{2} u_x^2 u_{2x}^2 + \frac{10}{21} u u_{2x}^2 - \frac{3}{14} u^2 u_{3x}^2 - \frac{5}{42} u_{2x}^3 u_{3x}^2 + \frac{1}{42} u u_{4x}^2 - \frac{1}{924} u_{5x}^2$</td>
</tr>
<tr>
<td>$\rho(8)$</td>
<td>$----$</td>
<td>$\frac{1}{8} u^8 - \frac{7}{5} u^5 u_x^2 - \frac{35}{12} u u_x^4 + \frac{7}{4} u^4 u_{2x}^2 + \frac{7}{3} u_x^2 u_{2x}^2 + \frac{5}{3} u^2 u_{2x}^3 + \frac{7}{24} u u_{2x}^4 + \frac{1}{2} u^3 u_{3x}^2 + \frac{1}{2} u_x^2 u_{3x}^2 - \frac{5}{6} u u_{2x} u_{3x}^2 + \frac{1}{12} u^2 u_{4x}^2 + \frac{1}{132} u_{2x}^2 u_{4x}^2 - \frac{1}{132} u u_{5x}^2 + \frac{1}{3432} u_{6x}^2$</td>
</tr>
</tbody>
</table>
Table 2: Conserved Densities for the Kaup-Kuperschmidt and Ito 5th-order equations

<table>
<thead>
<tr>
<th>Density</th>
<th>Kaup-Kuperschmidt equation</th>
<th>Ito equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_{(1)}$</td>
<td>$u$</td>
<td>$u$</td>
</tr>
<tr>
<td>$\rho_{(2)}$</td>
<td>$u_x$</td>
<td>$\frac{u^2}{2}$</td>
</tr>
<tr>
<td>$\rho_{(3)}$</td>
<td>$\frac{u^3}{3} - \frac{1}{8}u_x^2$</td>
<td>$-$</td>
</tr>
<tr>
<td>$\rho_{(4)}$</td>
<td>$\frac{u^4}{4} - \frac{9}{16}u u_x^2 + \frac{3}{64}u_x^4$</td>
<td>$\frac{u^4}{4} - \frac{9}{4}u u_x^2 + \frac{3}{4}u_x^2$</td>
</tr>
<tr>
<td>$\rho_{(5)}$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$\rho_{(6)}$</td>
<td>$\frac{u^6}{6} - \frac{35}{16} u^3 u_x^2 - \frac{31}{256} u_x^4 + \frac{51}{64} u^2 u_x^2$</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td>$+ \frac{37}{256} u_x^3 - \frac{15}{128} u u_x^3 + \frac{3}{512} u_x^4$</td>
<td></td>
</tr>
<tr>
<td>$\rho_{(7)}$</td>
<td>$\frac{u^7}{7} - \frac{27}{8} u^4 u_x^2 - \frac{369}{330} u u_x^4 + \frac{69}{40} u^3 u_x^2$</td>
<td>$-$</td>
</tr>
<tr>
<td></td>
<td>$+ \frac{2619}{4480} u_x^2 u_x^2 + \frac{2211}{2240} u u_x^3 - \frac{477}{1120} u_x^2 u_x^2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$- \frac{171}{640} u_x^3 u_x^2 + \frac{27}{360} u u_x^4 - \frac{9}{4480} u_x^2 u_x^2$</td>
<td></td>
</tr>
<tr>
<td>$\rho_{(8)}$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>Density</td>
<td>Sawada-Kotera-Ito equation</td>
<td>Lax equation</td>
</tr>
<tr>
<td>---------</td>
<td>----------------------------</td>
<td>--------------</td>
</tr>
<tr>
<td>ρ₁</td>
<td>u</td>
<td>u</td>
</tr>
<tr>
<td>ρ₂</td>
<td>−−−</td>
<td>u²</td>
</tr>
<tr>
<td>ρ₃</td>
<td>−u³ + uₓ²</td>
<td>−2u³ + uₓ²</td>
</tr>
<tr>
<td>ρ₄</td>
<td>3u⁴ − 9uuₓ² + u₂ₓ²</td>
<td>5u⁴ − 10uuₓ² + u₂ₓ²</td>
</tr>
<tr>
<td>ρ₅</td>
<td>−−−</td>
<td>−14u⁵ + 70u²uₓ² − 14uu₂ₓ² + u₃ₓ²</td>
</tr>
<tr>
<td>ρ₆</td>
<td>−12₇u⁶ + 150₇u³uₓ² + 17₇u⁴uₓ² − 4₇u²u₂ₓ² − 16₂₁u₂ₓ³ + uu₃ₓ² − 1₂₁u₄ₓ²</td>
<td>−7³u⁶ + 7₀₅u³uₓ² + 3₅₁₈u⁴uₓ² − 7u²u₂ₓ²</td>
</tr>
<tr>
<td>ρ₇</td>
<td>5u⁷ − 105u⁴uₓ² − 42uuₓ⁴ + 1₃₃₃u³u₂ₓ² + 2₁₄u₂ₓ⁴ + 2₉u²u₃ₓ² − 1₄₁₃u₆ₓ²</td>
<td>−2³u⁷ + 3₅₁₃u⁴uₓ² + 3₅₁₈uuₓ⁴ − 1₄₁₃u³u₂ₓ²</td>
</tr>
<tr>
<td>ρ₈</td>
<td>−−−</td>
<td>3²u⁸ − 4₂u⁵uₓ² − 3₅u²uₓ⁴ + 2₁u⁴u₂ₓ²</td>
</tr>
</tbody>
</table>

**Table 3: Conserved Densities for the Sawada-Kotera-Ito and Lax 7th-order equations**