Symbolic Computation of Scaling Invariant Lax Pairs in Operator Form for Integrable Systems

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Outline

• What are Lax pairs of nonlinear PDEs?
• Lax pairs in operator form
• Lax pairs in matrix form
• Reasons to compute Lax pairs
• Quick method to find Lax pairs
• More algorithmic approach
• Examples of Lax pairs of nonlinear PDEs
• Conclusions and future work
Peter D. Lax (1926-)

Seminal paper: Integrals of nonlinear equations of evolution and solitary waves
What are Lax Pairs of Nonlinear PDEs?

• Historical example: Korteweg-de Vries equation

\[ u_t + \alpha uu_x + u_{xxx} = 0 \]

• Key idea: Replace the nonlinear PDE with a compatible linear system (Lax pair):

\[
\begin{align*}
\psi_{xx} + \left( \frac{1}{6} \alpha u - \lambda \right) \psi &= 0 \\
\psi_t + 4\psi_{xxx} + \alpha u \psi_x + \frac{1}{2} \alpha u_x \psi + a(t) \psi &= 0
\end{align*}
\]

\( \psi \) is eigenfunction; \( \lambda \) is constant eigenvalue \((\lambda_t = 0)\) (isospectral), and \( a(t) \) is an arbitrary function. We will set \( a(t) = 0 \).
Class of Equations and Notation

• Consider a system of evolution equations:

\[ u_t = f(u, u_x, u_{xx}, \ldots, u_{Mx}) \]

with \( u(x, t) = (u^{(1)}, u^{(2)}, \ldots, u^{(N)}) \) and where

\[ u^{(j)}_{kx} = \frac{\partial^k u^{(j)}}{\partial x^k} \]

• In examples, the components of \( u \) are \( u, v, \ldots \)

• Define the total derivative operator as

\[ D_t \cdot = \frac{\partial \cdot}{\partial t} + \sum_{j=1}^{N} \sum_{k=0}^{M} \frac{\partial \cdot}{\partial u^{(j)}_{kx}} D_x^k \left( u^{(j)}_t \right) \]
Lax Pairs in Operator Form

• Replace a completely integrable nonlinear PDE by a pair of linear equations (called a Lax pair):

\[ \mathcal{L}\psi = \lambda\psi \quad \text{and} \quad D_t\psi = M\psi \]

• Require compatibility of both equations

\[ \mathcal{L}_t\psi + \mathcal{L}D_t\psi = \lambda D_t\psi \]
\[ \mathcal{L}_t\psi + \mathcal{L}M\psi = \lambda M\psi \]
\[ = M\lambda\psi \]
\[ \dot{=} M\mathcal{L}\psi \]

Hence,
\[ \mathcal{L}_t\psi + (\mathcal{L}M - M\mathcal{L})\psi \dot{=} 0 \]
• Lax equation: \( \mathcal{L}_t + [\mathcal{L}, \mathcal{M}] \dot{=} 0 \)

with commutator \([\mathcal{L}, \mathcal{M}] = \mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L}\).

Furthermore, \( \mathcal{L}_t \psi = [\mathcal{D}_t, \mathcal{L}] \psi = \mathcal{D}_t (\mathcal{L} \psi) - \mathcal{L} \mathcal{D}_t \psi \)

and \( \dot{=} \) means “evaluated on the PDE”

• Example: Lax operators for the KdV equation

\[
\mathcal{L} = \mathcal{D}_x^2 + \frac{1}{6} \alpha u \mathbf{I}
\]

\[
\mathcal{M} = - \left( 4 \mathcal{D}_x^3 + \alpha u \mathcal{D}_x + \frac{1}{2} \alpha u_x \mathbf{I} \right)
\]

• Note: \( \mathcal{L}_t \psi + [\mathcal{L}, \mathcal{M}] \psi = \frac{1}{6} \alpha \left( u_t + \alpha uu_x + u_{xxx} \right) \psi \)
Alternate Operator Formulations

• Define $\tilde{L} = L - \lambda I$ and $\tilde{M} = M - D_t$

• Then, the Lax pair becomes

$$\tilde{L}\psi = 0 \quad \text{and} \quad \tilde{M}\psi = 0$$

and the Lax equation becomes $[\tilde{L}, \tilde{M}] = O$

Challenge: Find commuting operators modulo the (nonlinear) PDE

• If $S$ is an arbitrary invertible operator, then

$$\hat{L} = SLS^{-1} \quad \hat{M} = SMS^{-1} \quad \hat{D}_t = S D_t S^{-1}$$

satisfy $\hat{L}_t + [\hat{L}, \hat{M}] = O$
Lax Pairs in Matrix Form

• Express compatibility of

\[ D_x \Psi = X \Psi \]
\[ D_t \Psi = T \Psi \]

where \( \Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{bmatrix} \), \( X \) and \( T \) are \( N \times N \) matrices

• Lax equation (zero-curvature equation):

\[ D_t X - D_x T + [X, T] \dot{=} 0 \]

with commutator \( [X, T] = XT - TX \)
• Example: Lax pair for the KdV equation

\[
X = \begin{bmatrix}
0 & 1 \\
\lambda - \frac{1}{6} \alpha u & 0
\end{bmatrix}
\]

\[
T = \begin{bmatrix}
\frac{1}{6} \alpha u_x & -4\lambda - \frac{1}{3} \alpha u \\
-4\lambda^2 + \frac{1}{3} \alpha \lambda u + \frac{1}{18} \alpha^2 u^2 + \frac{1}{6} \alpha u_{2x} & -\frac{1}{6} \alpha u_x
\end{bmatrix}
\]

Substitution into the Lax equation yields

\[
D_t X - D_x T + [X, T] = -\frac{1}{6} \alpha \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
u_t + \alpha uu_x + u_{3x} & 0
\end{bmatrix}
\]
Equivalence under Gauge Transformations

- Lax pairs are equivalent under a gauge transformation:

If $(X, T)$ is a Lax pair then so is $(\tilde{X}, \tilde{T})$ with

\[
\tilde{X} = G \, X \, G^{-1} + D_x(G) \, G^{-1}
\]
\[
\tilde{T} = G \, T \, G^{-1} + D_t(G) \, G^{-1}
\]

$G$ is arbitrary invertible matrix and $\tilde{\Psi} = G \Psi$.

Thus,

\[
\dot{\tilde{X}}_t - \tilde{T}_x + [\tilde{X}, \tilde{T}] = 0
\]
• Example: For the KdV equation

\[
\begin{bmatrix}
0 & 1 \\
\lambda - \frac{1}{6} \alpha u & 0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
-ik & \frac{1}{6} \alpha u \\
-1 & ik
\end{bmatrix}
\]

Here,

\[
\tilde{X} = G \ X \ G^{-1}
\quad \text{and} \quad
\tilde{T} = G \ T \ G^{-1}
\]

with

\[
G = \begin{bmatrix}
-i k & 1 \\
-1 & 0
\end{bmatrix}
\]

where \( \lambda = -k^2 \)
Reasons to Compute a Lax Pair

• Compatible linear system is the starting point for application of the IST and the Riemann-Hilbert method for boundary value problems
• Confirm the complete integrability of the PDE
• Zero-curvature representation of the PDE
• Compute conservation laws of the PDE
• Discover families of completely integrable PDEs

Question: How to find a Lax pair of a completely integrable PDE?

Answer: There is no completely systematic method
Dilation Invariance and Weights

• KdV equation is invariant under dilation symmetry

\[(x, t, u) \rightarrow (\kappa^{-1} x, \kappa^{-3} t, \kappa^2 u) = (\tilde{x}, \tilde{t}, \tilde{u})\]

where \(\kappa\) is an arbitrary parameter. Indeed,

\[u_t + \alpha uu_x + u_{xxx} = 0 \rightarrow \frac{1}{\kappa^5} (\tilde{u}_{\tilde{t}} + \alpha \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}) = 0\]

• The weight \(W\) of a variable is the exponent of \(\kappa\) in the symmetry. Thus, \(W(x) = -1, W(t) = -3\), or

\[W(D_x) = 1, \quad W(D_t) = 3, \quad W(u) = 2\]

• The total weight of the KdV equation is 5 because each monomial scales with \(\kappa^5\)
• The Lax operators for the KdV equation are scaling invariant.

Indeed,

\[ \mathcal{L} = D_x^2 + \frac{1}{6} \alpha u I \]

is uniform of weight 2.

\[ \mathcal{M} = - \left( 4D_x^3 + \alpha u D_x + \frac{1}{2} \alpha u_x I \right) \]

is uniform of weight 3

• Furthermore, \( \mathcal{L}\psi = \lambda \psi \) and \( D_t \psi = \mathcal{M} \psi \) are uniform in weight if \( W(\lambda) = W(\mathcal{L}) = 2 \) and \( W(\mathcal{M}) = W(D_t) = 3 \)
Elementary Method to Compute Lax Pairs

Using the KdV equation as an example

- Select $W(\mathcal{L}) = 2$. Here $W(\mathcal{M}) = 3$. In general, $W(\mathcal{L}) \geq W(u)$ and $W(\mathcal{M}) = W(D_t)$.

- Build $\mathcal{L}$ and $\mathcal{M}$ as linear combinations of scaling invariant terms with undetermined coefficients:

  $$\mathcal{L} = D_x^2 + c_1 u I$$

  $$\mathcal{M} = c_2 D_x^3 + c_3 u D_x + c_4 u_x I$$

- Substitute into $\mathcal{L}_t + [\mathcal{L}, \mathcal{M}] = O$, and replace $u_t$ by $-(\alpha uu_x + u_3x)$
• Set the coefficients of $D_x^2, D_x, \text{ and } I$ equal to zero

• Set the coefficients of like monomial terms in $u, u_x, u_{xx}$, etc. equal to zero

• Reduce the nonlinear algebraic system

\[
2c_3 - 3c_1c_2 = 0, \quad 2c_4 + c_3 - 3c_1c_2 = 0, \\
c_1(c_3 + \alpha) = 0, \quad c_1 - c_4 + c_1c_2 = 0
\]

with the Gröbner basis method into

\[
c_1(6c_1 - \alpha) = 0, \quad c_1(c_2 + 4) = 0, \quad c_1(c_3 + \alpha) = 0,
\]

\[
c_1(2c_4 + \alpha) = 0, \quad 6c_1 + c_3 = 0, \quad 3c_1 + c_4 = 0
\]

• Solve: $c_1 = \frac{1}{6}\alpha, \quad c_2 = -4, \quad c_3 = -\alpha, \quad c_4 = -\frac{1}{2}\alpha$
• Substitute the coefficients into $\mathcal{L}$ and $\mathcal{M}$:

$$\mathcal{L} = D_x^2 + \frac{1}{6} \alpha u I$$

$$\mathcal{M} = -\left(4D_x^3 + \alpha u D_x + \frac{1}{2} \alpha u_x I\right)$$

• In complicated cases the nonlinear algebraic systems are long and hard to solve (too many solution branches)

• A divide and conquer strategy is needed
Algorithm to Compute Lax Pairs
Using the KdV equation as an example

• Step 1: Compute the weights

\[ W(D_x) = 1, \quad W(D_t) = 3, \quad W(u) = 2 \]

• Step 2: Build a candidate Lax pair

Select \( W(L) = 2 \). Here \( W(M) = 3 \).

The candidate Lax pair is

\[
\begin{align*}
L &= D_x^2 + f_1 D_x + f_0 I \\
M &= c_3 D_x^3 + g_2 D_x^2 + g_1 D_x + g_0 I
\end{align*}
\]

with undetermined functions \( f_0, f_1, g_0, g_1, g_2 \) and undetermined constant coefficient \( c_3 \).
• **Step 3:** Substitute into the Lax equation

\[
\mathcal{L}_t + [\mathcal{L}, \mathcal{M}] = \\
\left( 2D_x g_2 - 3c_3 D_x f_1 \right) D_x^3 \\
+ \left( D_x^2 g_2 - 3c_3 D_x^2 f_1 + f_1 D_x g_2 + 2D_x g_1 - 2g_2 D_x f_1 \\
- 3c_3 D_x f_0 \right) D_x^2 \\
+ \left( D_t f_1 - c_3 D_x^3 f_1 + D_x^2 g_1 - g_2 D_x^2 f_1 - 3c_3 D_x^2 f_0 \\
+ f_1 D_x g_1 + 2D_x g_0 - g_1 D_x f_1 - 2g_2 D_x f_0 \right) D_x \\
+ \left( D_t f_0 - c_3 D_x^3 f_0 + D_x^2 g_0 - g_2 D_x^2 f_0 + f_1 D_x g_0 - g_1 D_x f_0 \right) I
\]
• Step 4: Solve the kinematic constraints (i.e., equations not involving $D_t$)

Equate the coefficients of $D^3_x$ and $D^2_x$ to zero and solve, yielding

\[ g_2 = \frac{3}{2} c_3 f_1, \]
\[ g_1 = \frac{3}{4} c_3 D_x f_1 + \frac{3}{8} c_3 f_1^2 + \frac{3}{2} c_3 f_0 \]

• The candidate $M$ operator reduces to

\[ M = c_3 D^3_x + \frac{3}{2} c_3 f_1 D^2_x + \frac{3}{8} c_3 \left( 2 D_x f_1 + f_1^2 + 4 f_0 \right) D_x + g_0 I \]

• The candidate $L$ remains unchanged
• Step 5: Solve the dynamical equations (i.e., equations that do involve $D_t$)

The coefficients of $I$ and $D_x$ yield

$$D_t f_1 + 2D_x g_0 - \frac{1}{8} c_3 D_x \left( 2D_x^2 f_1 + 12D_x f_0 - f_1^3 + 12f_1f_0 \right) = 0$$

$$D_t f_0 + D_x^2 g_0 + f_1 D_x g_0 - c_3 \left( D_x^3 f_0 + \frac{3}{2} f_1 D_x^2 f_0 + \frac{3}{4} D_x f_1 D_x f_0 + \frac{3}{8} f_1^2 D_x f_0 + \frac{3}{2} f_0 D_x f_0 \right) = 0$$

• Because $W(\mathcal{L}) = 2$ one has $f_1 = 0$. Thus,

$$2D_x g_0 - \frac{3}{2} c_3 D_x^2 f_0 = 0$$

$$D_t f_0 + D_x^2 g_0 - c_3 \left( D_x^3 f_0 + \frac{3}{2} f_0 D_x f_0 \right) = 0$$
• Step 5: continued

Solving these equations gives

\[ g_0 = \frac{3}{4} c_3 D_x f_0 \quad \text{and} \quad f_0 = b_0 u \]

• Replace \( u_t \) by \(- (\alpha uu_x + u_3x)\),

\[ \left( \alpha + \frac{3}{2} c_3 b_0 \right) uu_x + \left( 1 + \frac{1}{4} c_3 \right) u_3x = 0 \]

• Hence,

\[ c_3 = -4, \quad b_0 = \frac{1}{6} \alpha, \quad f_0 = \frac{1}{6} \alpha u, \quad f_1 = 0, \quad g_0 = -\frac{1}{2} \alpha u_x \]
Step 6: Substitute the coefficients into the undetermined functions and these into the candidate pair.

Thus,

\[ \mathcal{L} = D_x^2 + \frac{1}{6} \alpha u I \]

and

\[ \mathcal{M} = - \left( 4 D_x^3 + \alpha u D_x + \frac{1}{2} \alpha u_x I \right) \]

form a Lax pair for the KdV equation.
Algorithm for Computing Lax Pairs

- Compute the scaling symmetry of the PDE
- Select $W(\mathcal{L}) = l \geq 1$.
  
  From the Lax equation: $W(\mathcal{M}) = W(\partial_t) = m$

- Build a candidate Lax pair of the form

\[
\mathcal{L} = D_x^l + f_{l-1}D_x^{l-1} + \ldots + f_0 I
\]
\[
\mathcal{M} = c_m D_x^m + g_{m-1}D_x^{m-1} + \ldots + g_0 I
\]

for a constant $c_m$

- Substitute into the Lax equation
• Separate into **kinematic constraints and dynamical equations**
• Solve the kinematic equations
• Solve the dynamical equations
• Substitute the coefficients into undetermined functions and these into the candidate Lax pair
• Test the Lax pair
• Example 1: The modified KdV (mKdV) equation

\[ u_t + \alpha u^2 u_x + u_{3x} = 0 \]

has weights of \( W(u) = W(D_x) = 1 \) and \( W(D_t) = 3 \)

• Selecting \( W(L) = 1 \) gives a trivial Lax pair

• Select \( W(L) = 2 \), as in the KdV case, yields

\[
\mathcal{L} = D_x^2 + f_1 D_x + f_0 I \\
\mathcal{M} = c_3 D_x^3 + g_2 D_x^2 + g_1 D_x + g_0 I
\]

• Requiring uniform weights gives

\[
f_1 = b_0 u, \quad f_0 = b_1 u^2 + b_2 u_x, \quad g_0 = a_1 u^3 + a_2 uu_x + a_3 u_{xx}\]
Example 1: The mKdV equation – continued

Solving the kinematic constraints and dynamical equations gives the Lax pair

\[ L = D_x^2 + 2\epsilon u D_x + \frac{1}{6} \left( (6\epsilon^2 + \alpha) u^2 + (6\epsilon \pm \sqrt{-6\alpha}) u_x \right) \]

\[ M = -4D_x^3 - 12\epsilon u D_x^2 \]

\[- \left( (12\epsilon^2 + \alpha) u^2 + (12\epsilon \pm \sqrt{-6\alpha}) u_x \right) D_x \]

\[- \left( (4\epsilon^3 + \frac{2}{3}\epsilon\alpha) u^3 + (12\epsilon^2 \pm \epsilon\sqrt{-6\alpha} + \alpha) uu_x \right) \]

\[ + \left( 3\epsilon \pm \frac{1}{2}\sqrt{-6\alpha} \right) u_{xx} \]

• Example 2: The Boussinesq system

\[ u_t - v_x = 0 \]
\[ v_t - \beta u_x + 3uu_x + \alpha u_3x = 0 \]

has \( W(D_x) = 1, W(D_t) = W(u) = W(\beta) = 2, W(\nu) = 3 \)

• Select \( W(\mathcal{L}) = 3 \). Then,

\[ \mathcal{L} = D_x^3 + f_1D_x + f_0I \]
\[ \mathcal{M} = c_2D_x^2 + g_0I \]

• The kinematic constraint yields \( g_0 = \frac{2}{3} c_2 f_1 + c_0\beta \)

The dynamical equations then become

\[ D_tf_1 = c_2 \left( 2Dxf_0 - D_x^2f_1 \right) \]
\[ D_tf_0 = c_2 \left( D_x^2f_0 - \frac{2}{3}D_x^3f_1 - \frac{2}{3}f_1D_xf_1 \right) \]
• **Example 2: The Boussinesq system – continued**

• The uniform weight ansatz gives

\[
\begin{align*}
 f_1 &= a_1 u + a_2 \beta \\
f_0 &= a_3 u_x + D_x^{-1} \left( a_4 u^2 + a_5 \beta u + a_6 v_x + a_7 \beta^2 \right)
\end{align*}
\]

• Solving the dynamical equations gives

\[
\begin{align*}
 \mathcal{L} &= D_x^3 + \frac{1}{4\alpha} (3u - \beta) D_x + \frac{3}{8\alpha^2} \left( \alpha u_x \pm \frac{1}{3} \sqrt{3\alpha} v \right) I \\
 \mathcal{M} &= \pm \sqrt{3\alpha} D_x^2 \pm \frac{\sqrt{3\alpha}}{2\alpha} u I
\end{align*}
\]

[V. E. Zakharov, Sov. Phys. JETP, 1974]
• **Example 3:** The coupled KdV system (Hirota & Satsuma)

\[ u_t - 6\beta uu_x + 6vv_x - \beta u_3x = 0 \]
\[ v_t + 3uv_x + v_3x = 0 \]

has \( W(D_x) = 1, W(D_t) = 3, W(u) = W(v) = 2. \)

• **Select** \( W(L) = 4. \) If \( \beta = \frac{1}{2}, \) then

\[
L = D_x^4 + 2uD_x^2 + 2(u_x - v_x)D_x + (u^2 - v^2 + u_{2x} - v_{2x})I
\]
\[
M = 2D_x^3 + 3uD_x + 3\left(\frac{1}{2}u_x - v_x\right)I
\]

• Example 4: The Drinfel’d-Sokolov-Wilson system

\[ u_t + 3vv_x = 0, \quad v_t + 2uv_x + \alpha u_x v + 2v_3x = 0 \]

has \( W(D_x) = 1, W(D_t) = 3, W(u) = W(v) = 2. \)

• Select \( W(\mathcal{L}) = 6. \) If \( \alpha = 1, \) then

\[
\mathcal{L} = D_x^6 + 2uD_x^4 + (4u_x - 3v_x)D_x^3 \\
+ \left( \frac{9}{2} (u_{2x} - v_{2x}) - u^2 - v^2 \right) D_x^2 \\
+ \left( \frac{5}{2} (u_{3x} - v_{3x}) + 2 (uu_x - vv_x) + u_x v - uv_x \right) D_x \\
+ \left( \frac{1}{2} (u_{4x} - v_{4x}) + \frac{1}{2} (u + v)(u_{2x} - v_{2x}) + \frac{1}{4} (u_x^2 - v_x^2) \right) \mathcal{I}
\]

\[
\mathcal{M} = D_x^3 + uD_x - \frac{1}{2} (3v_x - u_x) \mathcal{I}
\]

Example 5: Class of fifth-order KdV equations

\[ u_t + \alpha u^2 u_x + \beta u_x u_{xx} + \gamma uu_{3x} + u_5u = 0 \]

includes several completely integrable equations:

<table>
<thead>
<tr>
<th>Parameter ratios</th>
<th>Commonly used values</th>
<th>Equation name</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \left( \frac{\alpha}{2}, \frac{\beta}{\gamma} \right) ]</td>
<td>(30, 20, 10), (120, 40, 20), (270, 60, 30)</td>
<td>Lax</td>
</tr>
<tr>
<td>[ \left( \frac{1}{5}, 1 \right) ]</td>
<td>(5, 5, 5), (180, 30, 30), (45, 15, 15)</td>
<td>Sawada-Kotera</td>
</tr>
<tr>
<td>[ \left( \frac{1}{5}, \frac{5}{2} \right) ]</td>
<td>(20, 25, 10)</td>
<td>Kaup-Kupershmidt</td>
</tr>
</tbody>
</table>
Example 5: Fifth-order equations – continued

For $W(\mathcal{L}) = 2$, only Lax’s equation has a Lax pair

$$\mathcal{L} = D_x^2 + \frac{1}{10} \gamma u I$$

$$\mathcal{M} = -16 D_x^5 - 4 \gamma u D_x^3 - 6 \gamma u_x D_x^2 - \gamma \left(5u_{xx} + \frac{3}{10} \gamma u^2\right) D_x$$

$$- \gamma \left(\frac{3}{2} u_{3x} + \frac{3}{10} \gamma uu_x\right) I$$

Example 5: Fifth-order equations – continued

For $W(\mathcal{L}) = 3$, the Sawada-Kotera and Kaup-Kupershmidt equations have Lax pairs

For the Kaup-Kupershmidt equation:

$$\mathcal{L} = D_x^3 + \frac{1}{5} \gamma u D_x + \frac{1}{10} \gamma u_x I$$

$$\mathcal{M} = 9 D_x^5 + 3 \gamma u D_x^3 + \frac{9}{2} \gamma u_x D_x^2 + \left( \frac{1}{5} \gamma^2 u^2 + \frac{7}{2} \gamma u_{xx} \right)$$

$$+ \left( \frac{1}{5} \gamma^2 uu_x + \gamma u_{3x} \right) I$$

• Example 5: Fifth-order equations — continued

• For the Sawada-Kotera equation with $W(\mathcal{L}) = 3$:

\[
\mathcal{L} = D_x^3 + \frac{1}{5} \gamma u D_x
\]

\[
\mathcal{M} = 9 D_x^5 + 3 \gamma u D_x^3 + 3 \gamma u_x D_x^2 + \left( \frac{1}{5} \gamma^2 u^2 + 2 \gamma u_{2x} \right) D_x
\]


Computations also resulted in:

\[
\tilde{\mathcal{L}} = D_x^3 + \frac{1}{5} \gamma u D_x + \frac{1}{5} \gamma u_x I = D_x \mathcal{L} D_x^{-1}
\]

\[
\tilde{\mathcal{M}} = 9 D_x^5 + 3 \gamma u D_x^3 + 6 \gamma u_x D_x^2 + \left( \frac{1}{5} \gamma^2 u^2 + 5 \gamma u_{2x} \right) D_x
\]

\[
+ \left( \frac{2}{5} \gamma^2 uu_x + 2 \gamma u_{3x} \right) I = D_x \mathcal{M} D_x^{-1}
\]
Conclusions and Future Work

- Scaling invariant Lax pairs in operator form are fairly easy to construct
- Scaling invariant Lax pairs in matrix form are hard to construct
- Gauge equivalence: Which Lax pairs are useful, which ones are not?
- Compare with Wahlquist & Estabrook method, pseudo-differential operator method, etc.
- Implementation in Mathematica
Thank You