Symbolic Computation of
Conserved Densities, Generalized Symmetries
and Recursion Operators for Nonlinear Evolution
and Lattice Equations

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OUTLINE

Purpose & Motivation

PART I: Partial Differential Equations (PDEs)
  • Key Concept and Definitions
  • Algorithm for Conservation Laws
  • Algorithm for Generalized Symmetries
  • Algorithm for Recursion Operators

PART II: Differential-difference Equations (DDEs)
  • Key Concept and Definitions
  • Algorithm for Conservation Laws
  • Algorithm for Generalized Symmetries

PART III: Software
  • Scope and Limitations of Algorithms
  • Implementation in Mathematica – Software
  • Other Packages
  • Conclusions & Future Research
  • Publications
• **Purpose**

Design and implement algorithms to compute polynomial conservation laws, symmetries, and recursion operators for nonlinear systems of evolution and lattice equations.

• **Motivation**

- Conservation laws describe the conservation of fundamental physical quantities (linear momentum, energy, etc.). Compare with constants of motion in mechanics.

- Conservation laws provide a method to study quantitative and qualitative properties of equations and their solutions, e.g. Hamiltonian structures.

- Conservation laws can be used to test numerical integrators.

- For nonlinear PDEs and lattices, the existence of a sufficiently large (in principal infinite) number of conservation laws or symmetries assures complete **integrability**.

- The existence of a recursion operator assures the existence of infinitely many symmetries.
PART I: Evolution Equations (PDEs)

• System of evolution equations

\[ u_t = F(u, u_x, u_{2x}, \ldots, u_{mx}) \]

in a (single) space variable \( x \) and time \( t \), and with

\[ u = (u_1, u_2, \ldots, u_n), \quad F = (F_1, F_2, \ldots, F_n). \]

Notation:

\[ u_{mx} = u^{(m)} = \frac{\partial u}{\partial x^m}. \]

\( F \) is polynomial in \( u, u_x, \ldots, u_{mx} \).

PDEs of higher order in \( t \) should be recast as a first-order system.

• Examples:

The Korteweg-de Vries (KdV) equation:

\[ u_t + uu_x + u_{3x} = 0. \]

Fifth-order evolution equations with constant parameters \( (\alpha, \beta, \gamma) \):

\[ u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma uu_{3x} + u_{5x} = 0. \]

Special case. The fifth-order Sawada-Kotera (SK) equation:

\[ u_t + 5u^2 u_x + 5u_x u_{2x} + 5uu_{3x} + u_{5x} = 0. \]

The Boussinesq (wave) equation:

\[ u_{tt} - u_{2x} + 3uu_{2x} + 3u_x^2 + \alpha u_{4x} = 0, \]

written as a first-order system (\( v \) auxiliary variable):

\[ u_t + v_x = 0, \]

\[ v_t + u_x - 3uu_x - \alpha u_{3x} = 0. \]
A vector nonlinear Schrödinger equation:

\[ \mathbf{B}_t + (|\mathbf{B}|^2 \mathbf{B})_x + (\mathbf{B}_0 \cdot \mathbf{B}_x) \mathbf{B}_0 + \mathbf{e} \times \mathbf{B}_{xx} = 0, \]

written in component form, \( \mathbf{B}_0 = (a, b) \) and \( \mathbf{B} = (u, v) \):

\[
\begin{align*}
    u_t &+ \left[ u(u^2 + v^2) + \beta u + \gamma v - v_x \right]_x = 0, \\
    v_t &+ \left[ v(u^2 + v^2) + \theta u + \delta v + u_x \right]_x = 0,
\end{align*}
\]

\[ \beta = a^2, \quad \gamma = \theta = ab, \quad \text{and} \quad \delta = b^2. \]

- **Key concept: Dilation invariance.**

Conservation laws, symmetries and recursion operators are invariant under the dilation (scaling) symmetry of the given PDE.

The KdV equation, \( u_t + uu_x + u_{3x} = 0 \), has scaling symmetry

\[ (t, x, u) \rightarrow (\lambda^{-3} t, \lambda^{-1} x, \lambda^2 u). \]

\( u \) corresponds to two \( x \)-derivatives, \( u \sim D_x^2 \). Similarly, \( D_t \sim D_x^3 \).

The weight, \( w \), of a variable equals the number of \( x \)-derivatives the variable carries.

Weights are rational. Weights of dependent variables are nonnegative.

Set \( w(D_x) = 1 \).

Due to dilation invariance: \( w(u) = 2 \) and \( w(D_t) = 3 \).

Consequently, \( w(x) = -1 \) and \( w(t) = -3 \).

The rank of a monomial is its total weight in terms of \( x \)-derivatives.
Every monomial in the KdV equation has rank 5. The KdV equation is *uniform in rank*.

What do we do if equations are not uniform in rank? Extend the space of dependent variables with parameters carrying weight.

Example: the Boussinesq system

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + \frac{\partial u}{\partial x} - 3uu_x - \alpha u_{3x} &= 0,
\end{align*}
\]

is not scaling invariant (\(u_x\) and \(u_{3x}\) are conflict terms). Introduce an auxiliary parameter \(\beta\)

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + \beta u_x - 3uu_x - \alpha u_{3x} &= 0,
\end{align*}
\]

which has scaling symmetry:

\[
(x, t, u, v, \beta) \rightarrow (\lambda x, \lambda^2 t, \lambda^{-2} u, \lambda^{-3} v, \lambda^{-2} \beta).
\]

- **CONSERVATION LAWS.**

\[
D_t \rho + D_x J = 0,
\]

with conserved density \(\rho\) and flux \(J\).

Both are polynomial in \(u, u_x, u_{2x}, u_{3x}, \ldots\)

\[
P = \int_{-\infty}^{+\infty} \rho \, dx = \text{constant}
\]

if \(J\) vanishes at infinity.

Conserved densities are equivalent if they differ by a \(D_x\) term.
**Example:** The Korteweg-de Vries (KdV) equation

\[ u_t + uu_x + u_{3x} = 0. \]

Conserved densities:

\[ \rho_1 = u, \quad D_t(u) + D_x \left( \frac{u^2}{2} + u_{2x} \right) = 0. \]

\[ \rho_2 = u^2, \quad D_t(u^2) + D_x \left( \frac{2u^3}{3} + 2uu_{2x} - u_x^2 \right) = 0. \]

\[ \rho_3 = u^3 - 3u_x^2, \]

\[ D_t \left( u^3 - 3u_x^2 \right) + D_x \left( \frac{3}{4}u^4 - 6uu_x^2 + 3u^2u_{2x} + 3u_x^2 - 6u_xu_{3x} \right) = 0. \]

\[ \vdots \]

\[ \rho_6 = u^6 - 60u^3u_x^2 - 30u_x^4 + 108u^2u_{2x}^2 \]

\[ + \frac{720}{7}u_{2x}^3 - \frac{648}{7}uu_{3x}^2 + \frac{216}{7}u_{4x}^2. \]

Time and space dependent conservation law:

\[ D_t \left( tu^2 - 2xu \right) \]

\[ + D_x \left( \frac{2}{3}tu^3 - xu^2 + 2tuu_{2x} - tu_x^2 - 2xu_{2x} + 2u_x \right) = 0. \]

**Algorithm for Conservation Laws of PDEs.**

1. Determine weights (scaling properties) of variables and auxiliary parameters.

2. Construct the form of the density (find monomial building blocks).

3. Determine the constant coefficients.
Example: Density of rank 6 for the KdV equation.

Step 1: Compute the weights.

 Require uniformity in rank. With \( w(D_x) = 1 \):

\[
w(u) + w(D_t) = 2w(u) + 1 = w(u) + 3.\]

Solve the linear system: \( w(u) = 2 \), \( w(D_t) = 3 \).

Step 2: Determine the form of the density.

List all possible powers of \( u \), up to rank 6 : \( [u, u^2, u^3] \).

Introduce \( x \) derivatives to ‘complete’ the rank.

\( u \) has weight 2, introduce \( D_x^4 \).

\( u^2 \) has weight 4, introduce \( D_x^2 \).

\( u^3 \) has weight 6, no derivative needed.

Apply the \( D_x \) derivatives.
Remove terms of the form \( D_x u_{px} \), or \( D_x \) up to terms kept prior in the list.

\[
[u_{4x}] \rightarrow [] \quad \text{empty list}.
\]

\[
[u_x^2, uu_{2x}] \rightarrow [u_x^2] \quad \text{since } uu_{2x} = (uu_x)_x - u_x^2.
\]

\[
[u^3] \rightarrow [u^3].
\]

Linearly combine the ‘building blocks’:

\[
\rho = c_1 u^3 + c_2 u_x^2.
\]
Step 3: Determine the coefficients \( c_i \).

Compute \( D_t \rho = 3c_1 u^2 u_t + 2c_2 u_x u_{xt} \).

Replace \( u_t \) by \(-(uu_x + u_3 x)\) and \( u_{xt} \) by \(-(uu_x + u_3 x)_x\).

Integrate the result, \( E \), with respect to \( x \). To avoid integration by parts, apply the Euler operator (variational derivative)

\[
L_u = \sum_{k=0}^{m} (-D_x)^k \frac{\partial}{\partial u_{kx}}
\]

\[
= \frac{\partial}{\partial u} - D_x (\frac{\partial}{\partial u_x}) + D_x^2 (\frac{\partial}{\partial u_{2x}}) + \cdots + (-1)^m D_x^m (\frac{\partial}{\partial u_{mx}}).
\]

to \( E \) of order \( m \).

If \( L_u(E) = 0 \) immediately, then \( E \) is a total \( x \)-derivative.

If \( L_u(E) \neq 0 \), the remaining expression must vanish identically.

\[
D_t \rho = -D_x [\frac{3}{4} c_1 u^4 - (3c_1 - c_2) uu_x^2 + 3c_1 u^2 u_{2x} - c_2 u_{2x}^2 + 2c_2 u_x u_{3x}] - (3c_1 + c_2) u_x^3.
\]

The non-integrable term must vanish.

So, \( c_1 = -\frac{1}{3} c_2 \). Set \( c_2 = -3 \), hence, \( c_1 = 1 \).

Result:

\[
\rho = u^3 - 3u_x^2.
\]

Expression \([\ldots]\) yields

\[
J = \frac{3}{4} u^4 - 6uu_x^2 + 3u^2 u_{2x} + 3u_{2x}^2 - 6u_x u_{3x}.
\]

Example: First few densities for the Boussinesq system:

\[
\rho_1 = u, \quad \rho_2 = v, \quad \rho_3 = uv, \quad \rho_4 = \beta u^2 - u^3 + v^2 + \alpha u_x^2.
\]

(then substitute \( \beta = 1 \)).
Application.

A Class of Fifth-Order Evolution Equations

\[ u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma uu_{3x} + u_{5x} = 0 \]

where \( \alpha, \beta, \gamma \) are nonzero parameters.

\[ u \sim D_x^2. \]

Special cases:

- \( \alpha = 30 \quad \beta = 20 \quad \gamma = 10 \) Lax.
- \( \alpha = 5 \quad \beta = 5 \quad \gamma = 5 \) Sawada – Kotera.
- \( \alpha = 20 \quad \beta = 25 \quad \gamma = 10 \) Kaup – Kupershmidt.
- \( \alpha = 2 \quad \beta = 6 \quad \gamma = 3 \) Ito.

What are the conditions for the parameters \( \alpha, \beta \) and \( \gamma \) so that the equation admits a density of fixed rank?

- **Rank 2:**
  No condition
  \[ \rho = u. \]

- **Rank 4:**
  Condition: \( \beta = 2\gamma \) (Lax and Ito cases)
  \[ \rho = u^2. \]
- **Rank 6:**
  Condition:
  \[
  10\alpha = -2\beta^2 + 7\beta\gamma - 3\gamma^2
  \]
  (Lax, SK, and KK cases)
  \[
  \rho = u^3 + \frac{15}{(2\beta + \gamma)}u_x^2.
  \]

- **Rank 8:**

  1. \(\beta = 2\gamma\) (Lax and Ito cases)
     \[
     \rho = u^4 - \frac{6\gamma}{\alpha}uu_x^2 + \frac{6}{\alpha}u_{2x}^2.
     \]
  
  2. \(\alpha = \frac{-2\beta^2 - 7\beta\gamma - 4\gamma^2}{45}\) (SK, KK and Ito cases)
      \[
      \rho = u^4 - \frac{135}{2\beta + \gamma}uu_x^2 + \frac{675}{(2\beta + \gamma)^2}u_{2x}^2.
      \]

- **Rank 10:**

  Condition:
  \[
  \beta = 2\gamma
  \]
  and
  \[
  10\alpha = 3\gamma^2
  \]
  (Lax case)
  \[
  \rho = u^5 - \frac{50}{\gamma}u^2u_x^2 + \frac{100}{\gamma^2}uu_{2x}^2 - \frac{500}{7\gamma^3}u_{3x}^2.
  \]
What are the necessary conditions for the parameters $\alpha$, $\beta$ and $\gamma$ so that the equation admits $\infty$ many polynomial conservation laws?

- If $\alpha = \frac{3}{10} \gamma^2$ and $\beta = 2\gamma$ then there is a sequence (without gaps!) of conserved densities (Lax case).

- If $\alpha = \frac{1}{5} \gamma^2$ and $\beta = \gamma$ then there is a sequence (with gaps!) of conserved densities (SK case).

- If $\alpha = \frac{1}{5} \gamma^2$ and $\beta = \frac{5}{2} \gamma$ then there is a sequence (with gaps!) of conserved densities (KK case).

- If
  
  \[ \alpha = -\frac{2\beta^2 - 7\beta\gamma + 4\gamma^2}{45} \]

  or
  
  \[ \beta = 2\gamma \]

  then there is a conserved density of rank 8.

Combine both conditions: $\alpha = \frac{2\gamma^2}{9}$ and $\beta = 2\gamma$ (Ito case).
**GENERALIZED SYMMETRY.**

\[
\mathbf{G}(x, t, \mathbf{u}, \mathbf{u}_x, \mathbf{u}_{2x}, \ldots)
\]

with \( \mathbf{G} = (G_1, G_2, \ldots, G_n) \) is a *symmetry* iff it leaves the PDE invariant for the replacement \( \mathbf{u} \to \mathbf{u} + \epsilon \mathbf{G} \) within order \( \epsilon \). i.e.

\[
\mathcal{D}_t(\mathbf{u} + \epsilon \mathbf{G}) = \mathbf{F}(\mathbf{u} + \epsilon \mathbf{G})
\]

must hold up to order \( \epsilon \) on the solutions of PDE.

Consequently, \( \mathbf{G} \) must satisfy the linearized equation

\[
\mathcal{D}_t \mathbf{G} = \mathbf{F}'(\mathbf{u})[\mathbf{G}],
\]

where \( \mathbf{F}' \) is the Fréchet derivative of \( \mathbf{F} \), i.e.,

\[
\mathbf{F}'(\mathbf{u})[\mathbf{G}] = \frac{\partial}{\partial \epsilon} \mathbf{F}(\mathbf{u} + \epsilon \mathbf{G})|_{\epsilon=0}.
\]

Here \( \mathbf{u} \) is replaced by \( \mathbf{u} + \epsilon \mathbf{G} \), and \( \mathbf{u}_{nx} \) by \( \mathbf{u}_{nx} + \epsilon \mathcal{D}_x^n \mathbf{G} \).

**Example.**

Consider the KdV equation

\[
u_t = 6\mathbf{u}\mathbf{u}_x + \mathbf{u}_{3x}.
\]

Generalized symmetries:

\[
\begin{align*}
\mathbf{G}^{(1)} &= \mathbf{u}_x, \\
\mathbf{G}^{(2)} &= 6\mathbf{u}\mathbf{u}_x + \mathbf{u}_{3x}, \\
\mathbf{G}^{(3)} &= 30\mathbf{u}^2\mathbf{u}_x + 20\mathbf{u}_x\mathbf{u}_{2x} + 10\mathbf{u}\mathbf{u}_{3x} + \mathbf{u}_{5x}, \\
\mathbf{G}^{(4)} &= 140\mathbf{u}^3\mathbf{u}_x + 70\mathbf{u}_x^3 + 280\mathbf{u}\mathbf{u}_x\mathbf{u}_{2x} + 70\mathbf{u}^2\mathbf{u}_{3x} \\
&+ 70\mathbf{u}_x\mathbf{u}_{3x} + 42\mathbf{u}_x\mathbf{u}_{4x} + 14\mathbf{u}\mathbf{u}_{5x} + \mathbf{u}_{7x}.
\end{align*}
\]
Algorithm for Generalized Symmetries of PDEs.

Consider the KdV equation, \( u_t = 6uu_x + u_{3x} \), with \( w(u) = 2 \).

**Step 1: Construct the form of the symmetry.**

Compute the form of the symmetry with rank 7.

List all powers in \( u \) with rank 7 or less:

\[ \mathcal{L} = \{1, u, u^2, u^3\} . \]

For each monomial in \( \mathcal{L} \), introduce the needed \( x \)-derivatives, so that each term exactly has rank 7. Thus,

\[
\begin{align*}
D_x(u^3) &= 3u^2u_x, \\
D_x^3(u^2) &= 6u_xu_{2x} + 2uu_{3x}, \\
D_x^5(u) &= u_{5x}, \\
D_x^7(1) &= 0.
\end{align*}
\]

Gather the resulting (non-zero) terms

\[ \mathcal{R} = \{u^2u_x, uu_xu_{2x}, uu_{3x}, u_{5x}\} . \]

The symmetry is a linear combination of these monomials:

\[ G = c_1 u^2u_x + c_2 uu_xu_{2x} + c_3 uu_{3x} + c_4 u_{5x} . \]

**Step 2: Determine the unknown coefficients \( c_i \).**

Compute \( D_tG \) and use KdV to remove \( u_t, u_{tx}, u_{txx}, \) etc.

Compute the Fréchet derivative.

Equate the resulting expressions.

Group the terms:

\[
(12c_1 - 18c_2)u_x^2u_{2x} + (6c_1 - 18c_3)uu_{2x}^2 + (6c_1 - 18c_3)uu_xu_{3x} + (3c_2 - 60c_4)u_{3x}^2 + (3c_2 + 3c_3 - 90c_4)u_{2x}u_{4x} + (3c_3 - 30c_4)u_xu_{5x} \equiv 0 .
\]
Solve the linear system:

\[ S = \{ 12c_1 - 18c_2 = 0, 6c_1 - 18c_3 = 0, 3c_2 - 60c_4 = 0, \\
3c_2 + 3c_3 - 90c_4 = 0, 3c_3 - 30c_4 = 0 \}. \]

Solution: \( \frac{c_1}{30} = \frac{c_2}{20} = \frac{c_3}{10} = c_4. \)

Setting \( c_4 = 1 \) one gets: \( c_1 = 30, c_2 = 20, c_3 = 10. \)

Substitute the result into the symmetry:

\[ G = 30u^2u_x + 20u_xu_{2x} + 10uu_{3x} + u_{5x}. \]

Note that \( u_t = G \) is known as the Lax equation.

- **x-t Dependent symmetries.**

The KdV equation has also symmetries which explicitly depend on \( x \) and \( t. \)

The same algorithm can be used provided the highest degree of \( x \) or \( t \) is specified.

Compute the symmetry of rank 2, that is linear in \( x \) or \( t. \)

List all monomials in \( u, x \) and \( t \) of rank 2 or less:

\[ \mathcal{L} = \{ 1, u, x, xu, t, tu, tu^2 \}. \]

For each monomial in \( \mathcal{L}. \) introduce enough \( x \)-derivatives, so that each term exactly has rank 2. Thus,

\[ D_x(xu) = u + xu_x, \quad D_x(tu^2) = 2tuu_x, \quad D^3_x(tu) = tu_{3x}, \]

\[ D^2_x(1) = D^2_x(x) = D^5_x(t) = 0. \]

Gather the non-zero resulting terms:

\[ \mathcal{R} = \{ u, xu_x, tuu_x, tu_{3x} \}, \]

Build the linear combination

\[ G = c_1 u + c_2 xu_x + c_3 tuu_x + c_4 tu_{3x}. \]
Determine the coefficients $c_1$ through $c_4$:

$$G = \frac{2}{3} u + \frac{1}{3} xu_x + 6tuu_x + tuu_x.$$ 

Two symmetries of KdV that explicitly depend on $x$ and $t$:

$$G = 1 + 6tu_x, \text{ and } G = 2u + xu_x + 3t(6uu_x + u_{3x}),$$

of rank 0 and 2, respectively.

**RECURSION OPERATORS.**

A *recursion operator* for a PDE system is the linear operator $\Phi$ connecting two symmetries $G$ and $\hat{G}$:

$$\hat{G} = \Phi G.$$

For $n$-component systems, $\Phi$ is an $n \times n$ matrix.

Defining equation for $\Phi$:

$$D_t \Phi + [\Phi, F'(u)] = \frac{\partial \Phi}{\partial t} + \Phi'[F] + \Phi \circ F'(u) - F'(u) \circ \Phi = 0,$$

where $[ , ]$ means commutator, $\circ$ stands for composition, and $\Phi'[F]$ is the variational derivative of $\Phi$.

**Example.**

The recursion operator for the KdV equation (has rank 2)

$$\Phi = D_x^2 + 2u + 2D_xuD_x^{-1} = D_x^2 + 4u + 2u_xD_x^{-1},$$

where $D_x^{-1}$ is the integration operator.

For example

$$\Phi u_x = (D_x^2 + 2u + 2D_xuD_x^{-1})u_x = 6uu_x + u_{3x},$$

$$\Phi(6uu_x + u_{3x}) = (D_x^2 + 2u + 2D_xuD_x^{-1})(6uu_x + u_{3x})$$

$$= 30u^2 u_x + 20uu_xu_{2x} + 10uu_{3x} + u_{5x}.$$
**Key Observations.**

The terms in the recursion operator are monomials in $D_x, D^{-1}_x, u, uu_x, \ldots$

Recursion operators split naturally in $\Phi = \Phi_0 + \Phi_1$.

$\Phi_0$ is a differential operator (no $D^{-1}_x$ terms).

$\Phi_1$ is an integral operator (with $D^{-1}_x$ terms).

Application of $\Phi$ to a symmetry should not leave any integrals.

For instance, for the KdV equation:

$D^{-1}_x(6uu_x + u_{3x}) = 3u^2 + u_{2x}$ is polynomial.

Use the conserved densities: $\rho^{(1)} = u, \rho^{(2)} = u^2, \rho^{(3)} = u^3 - \frac{1}{2}u_x^2$

$D_t\rho^{(1)} = D_t u = u_t = -D_x J^{(1)}$,  
$D_t\rho^{(2)} = D_t u^2 = 2uu_t = -D_x J^{(2)}$, and  
$D_t\rho^{(3)} = D_t(u^3 - \frac{1}{2}u_x^2) = \rho^{(3)'}(u)[u_t] = (3u^2 - u_x D_x)u_t = -D_x J^{(3)}$,

for polynomial $J^{(i)}, i = 1, 2, 3$.

So, application of $D^{-1}_x$, or $D^{-1}_x u$, or $D^{-1}_x(3u^2 - u_x D_x)$

to $6uu_x + u_{3x}$ leads to a polynomial result.

**Algorithm for Recursion Operators of PDEs.**

**Step 1: Determine the rank of the recursion operator.**

Recall: symmetries for the KdV equation, $u_t = 6uu_x + u_{3x}$, are

$G^{(1)} = u_x$,  
$G^{(2)} = 6uu_x + u_{3x}$,  
$G^{(3)} = 30u^2u_x + 20u_x u_{2x} + 10uu_{3x} + u_{5x}$.

Hence,

$R = \text{rank } \Phi = \text{rank } G^{(3)} - \text{rank } G^{(2)} = \text{rank } G^{(2)} - \text{rank } G^{(1)} = 2.$
Step 2: Construct the form of the recursion operator.

(i) Determine the pieces of operator $Φ_0$

List all permutations of type $D^j u^k$ of rank $R$, with $j$ and $k$ non-negative integers.

$$\mathcal{L} = \{D^2, u\}.$$ 

(ii) Determine the pieces of operator $Φ_1$

Combine the symmetries $G^{(j)}$ with $D^{-1}$ and $ρ^{(k)}(u)$, so that every term in

$$Φ_1 = \sum_j \sum_k G^{(j)}D^{-1}ρ^{(k)}(u)$$

has rank $Φ_1 = R$.

The indices $j$ and $k$ are taken so that

$$\text{rank} \left( G^{(j)} \right) + \text{rank} \left( ρ^{(k)}(u) \right) - 1 = R.$$

List such terms:

$$\mathcal{M} = \{u_xD^{-1}\}.$$ 

(iii) Build the operator $Φ$

Linearly combine the term in

$$\mathcal{R} = \mathcal{L} \cup \mathcal{M} = \{D^2, u, u_xD^{-1}\}.$$ 

to get

$$Φ = c_1 D^2 + c_2 u + c_3 u_xD^{-1}.$$
Step 3: Determine the unknown coefficients.

Require that
\[ \Phi G^{(k)} = G^{(k+1)}, \quad k = 1, 2, 3, \ldots \]

Solve the linear system:
\[ S = \begin{cases} 
  c_1 - 1 = 0, 
  18c_1 + c_3 - 20 = 0, 
  6c_1 + c_2 - 10 = 0, 
  2c_2 + c_3 - 10 = 0 
\end{cases}, \]

Solution: \( c_1 = 1, c_2 = 4, \) and \( c_3 = 2. \) So,
\[ \Phi = D^2 + 4u + 2u_x D^{-1}. \]

Examples.

The SK equation:
\[ u_t = 5u^2 u_x + 5u_x u_{2x} + 5uu_{3x} + u_{5x}. \]

Recursion operator:
\[
\Phi = D^6 + 3uD^4 - 3DuD^3 + 11D^2 uD^2 - 10D^3 uD + 5D^4 u \\
+ 12u^2 D^2 - 19uD uD + 8uD^2 u + 8DuDu + 4u^3 \\
+ u_x D^{-1} (u^2 - 2u_x D) + G^{(2)} D^{-1},
\]
with \( G^{(2)} = 5u^2 u_x + 5u_x u_{2x} + 5uu_{3x} + u_{5x}. \)

For the vector nonlinear Schrödinger system:
\[
\begin{align*}
  u_t + [u(u^2 + v^2) + \beta u + \gamma v - v_x]_x &= 0, \\
  v_t + [v(u^2 + v^2) + \theta u + \delta v + u_x]_x &= 0.
\end{align*}
\]

Recursion operator:
\[
\Phi = \begin{pmatrix}
  \beta - \delta + 2u^2 + 2u_x D^{-1} u & \theta + 2uv - D + 2u_x D^{-1} v \\
  \theta + 2uv + D + 2v_x D^{-1} u & 2v^2 + 2v_x D^{-1} v
\end{pmatrix}.
\]
PART II: Differential-difference (lattice) Equations

• Systems of lattices equations

Consider the system of lattice equations, continuous in time, discretized in (one dimensional) space

\[ \dot{u}_n = F(..., u_{n-1}, u_n, u_{n+1}, ...) \]

where \( u_n \) and \( F \) are vector dynamical variables.

\( F \) is polynomial with constant coefficients.

No restrictions on the level of the shifts or the degree of nonlinearity.

• CONSERVATION LAW:

\[ \dot{\rho}_n = J_n - J_{n+1} \]

with density \( \rho_n \) and flux \( J_n \).

Both are polynomials in \( u_n \) and its shifts.

\[ \frac{d}{dt}(\sum_n \rho_n) = \sum_n \dot{\rho}_n = \sum_n (J_n - J_{n+1}) \]

if \( J_n \) is bounded for all \( n \).

Subject to suitable boundary or periodicity conditions

\[ \sum_n \rho_n = \text{constant}. \]
• Example.

Consider the one-dimensional Toda lattice

\[ \ddot{y}_n = \exp(y_{n-1} - y_n) - \exp(y_n - y_{n+1}) \]

\( y_n \) is the displacement from equilibrium of the \( n \)th particle with unit mass under an exponential decaying interaction force between nearest neighbors.

Change of variables:

\[ u_n = \dot{y}_n, \quad v_n = \exp(y_n - y_{n+1}) \]

yields

\[ \dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}). \]

Toda system is completely integrable.

The first two density-flux pairs (computed by hand):

\[ \rho_n^{(1)} = u_n, \quad J_n^{(1)} = v_{n-1}, \quad \text{and} \quad \rho_n^{(2)} = \frac{1}{2}u_n^2 + v_n, \quad J_n^{(2)} = u_nv_{n-1}. \]

• Key concept: Dilation invariance.

The Toda system as well as the conservation laws and symmetries are invariant under the dilation symmetry

\[ (t, u_n, v_n) \rightarrow (\lambda^{-1}t, \lambda u_n, \lambda^2 v_n). \]

Thus, \( u_n \) corresponds to one \( t \)-derivative: \( u_n \sim \frac{d}{dt} \).

Similarly, \( v_n \sim \frac{d^2}{dt^2} \).

Weight, \( w \), of variables are defined in terms of \( t \)-derivatives.

Set \( w(\frac{d}{dt}) = 1 \).
Weights of dependent variables are nonnegative, rational, and independent of $n$.

Due to dilation invariance: $w(u_n) = 1$ and $w(v_n) = 2$.

The rank of a monomial is its total weight in terms of $t$-derivatives.

Require uniformity in rank for each equation to compute the weights:
(solve the linear system):

$$w(u_n) + 1 = w(v_n), \quad w(v_n) + 1 = w(u_n) + w(v_n),$$

Solving the linear system yields $w(u_n) = 1$, $w(v_n) = 2$.

- **Equivalence Criterion.**

Define $D$ shift-down operator, and $U$ shift-up operator, on the set of all monomials in $u_n$ and their shifts.

For a monomial $m$:

$$Dm = m|_{n \rightarrow n-1}, \quad \text{and} \quad Um = m|_{n \rightarrow n+1}.$$  

For example

$$Du_{n+2}v_n = u_{n+1}v_{n-1}, \quad Uu_{n-2}v_{n-1} = u_{n-1}v_n.$$  

Compositions of $D$ and $U$ define an equivalence relation. 
All shifted monomials are equivalent.

For example

$$u_{n-1}v_{n+1} \equiv u_{n+2}v_{n+4} \equiv u_{n-3}v_{n-1}.$$
Equivalence criterion:

Two monomials $m_1$ and $m_2$ are equivalent, $m_1 \equiv m_2$, if

$$m_1 = m_2 + [M_n - M_{n+1}]$$

for some polynomial $M_n$.

For example, $u_{n-2}u_n \equiv u_{n-1}u_{n+1}$ since

$$u_{n-2}u_n = u_{n-1}u_{n+1} + [u_{n-2}u_n - u_{n-1}u_{n+1}] = u_{n-1}u_{n+1} + [M_n - M_{n+1}].$$

Main representative of an equivalence class is the monomial with label $n$ on $u$ (or $v$).

For example, $u_n u_{n+2}$ is the main representative of the class with elements $u_{n-1}u_{n+1}, u_{n+1}u_{n+3}$, etc.

Use lexicographical ordering to resolve conflicts.

For example, $u_n v_{n+2}$ (not $u_{n-2}v_n$) is the main representative of the class with elements $u_{n-3}v_{n-1}, u_{n+2}v_{n+4}$, etc.

- **Steps of the Algorithm for Lattices.**

Three-step algorithm to find conserved densities:

1. Determine the weights.
2. Construct the form of density.
3. Determine the coefficients.
Example: Density of rank 3 or the Toda lattice,
\[ \dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}). \]

Step 1: Compute the weights.
Here \( w(u_n) = 1 \) and \( w(v_n) = 2 \).

Step 2: Construct the form of the density.
List all monomials in \( u_n \) and \( v_n \) of rank 3 or less:
\[ G = \{ u_n^3, u_n^2, u_n v_n, u_n, v_n \}. \]
For each monomial in \( G \), introduce enough \( t \)-derivatives to obtain weight 3. Use the lattice to remove \( \dot{u}_n \) and \( \dot{v}_n \):
\[ \frac{d^0}{dt^0}(u_n^3) = u_n^3, \quad \frac{d^0}{dt^0}(u_n v_n) = u_n v_n, \]
\[ \frac{d}{dt}(u_n^2) = 2u_n v_{n-1} - 2u_n v_n, \]
\[ \frac{d}{dt}(v_n) = u_n v_n - u_{n+1} v_n, \]
\[ \frac{d^2}{dt^2}(u_n) = u_{n-1} v_{n-1} - u_n v_{n-1} - u_n v_n + u_{n+1} v_n. \]
Gather the resulting terms in a set
\[ H = \{ u_n^3, u_n v_{n-1}, u_n v_n, u_{n-1} v_{n-1}, u_{n+1} v_n \}. \]
Replace members in the same equivalence class by their main representatives.
For example, \( u_n v_{n-1} \equiv u_{n+1} v_n \) are replaced by \( u_n v_{n-1} \).

Linearly combine the monomials in \( I = \{ u_n^3, u_n v_{n-1}, u_n v_n \} \) to obtain
\[ \rho_n = c_1 u_n^3 + c_2 u_n v_{n-1} + c_3 u_n v_n. \]
Step 3: Determine the coefficients.

Require that $\dot{\rho}_n = J_n - J_{n+1}$, holds.

Compute $\dot{\rho}_n$ and use the lattice to remove $\dot{u}_n$ and $\dot{v}_n$.

Group the terms

$$\dot{\rho}_n = (3c_1 - c_2)u_n^2v_{n-1} + (c_3 - 3c_1)u_n^2v_n + (c_3 - c_2)v_{n-1}v_n$$

$$+ c_2u_{n-1}u_nv_{n-1} + c_2v_{n-1}^2 - c_3u_nu_{n+1}v_n - c_3v_n^2.$$  

Use the equivalence criterion to modify $\dot{\rho}_n$.

Replace $u_{n-1}u_nv_{n-1}$ by $u_nu_{n+1}v_n + [u_{n-1}u_nv_{n-1} - u_nu_{n+1}v_n]$.

Introduce the main representatives. Thus

$$\dot{\rho}_n = (3c_1 - c_2)u_n^2v_{n-1} + (c_3 - 3c_1)u_n^2v_n$$

$$+ (c_3 - c_2)v_nv_{n+1} + [(c_3 - c_2)v_{n-1}v_n - (c_3 - c_2)v_nv_{n+1}]$$

$$+ c_2u_{n-1}u_nv_{n-1} + [c_2u_{n-1}u_nv_{n-1} - c_2u_nu_{n+1}v_n]$$

$$+ c_2v_n^2 + [c_2v_{n-1}^2 - c_2v_n^2] - c_3u_nu_{n+1}v_n - c_3v_n^2.$$  

Group the terms outside of the square brackets and move the pairs inside the square brackets to the bottom.

Rearrange the terms to match the pattern $[J_n - J_{n+1}]$.

Hence

$$\dot{\rho}_n = (3c_1 - c_2)u_n^2v_{n-1} + (c_3 - 3c_1)u_n^2v_n$$

$$+ (c_3 - c_2)v_nv_{n+1} + (c_2 - c_3)u_nu_{n+1}v_n + (c_2 - c_3)v_n^2$$

$$+ \{(c_3 - c_2)v_{n-1}v_n + c_2u_{n-1}u_nv_{n-1} + c_2v_{n-1}^2\}$$

$$- \{(c_3 - c_2)v_nv_{n+1} + c_2u_nv_{n+1}v_n + c_2v_n^2\}.$$  

The terms inside the square brackets determine:

$$J_n = (c_3 - c_2)v_{n-1}v_n + c_2u_{n-1}u_nv_{n-1} + c_2v_{n-1}^2.$$
The terms outside the square brackets must vanish, thus

\[ S = \{3c_1 - c_2 = 0, c_3 - 3c_1 = 0, c_2 - c_3 = 0\}. \]

The solution is \(3c_1 = c_2 = c_3\), so choose \(c_1 = \frac{1}{3}\), and \(c_2 = c_3 = 1\):

\[ \rho_n = \frac{1}{3} u_n^3 + u_n(v_{n-1} + v_n), \quad J_n = u_{n-1}u_nv_{n-1} + v_{n-1}^2. \]

Analogously, conserved densities of rank \(\leq 5\):

\[ \rho_n^{(1)} = u_n \quad \rho_n^{(2)} = \frac{1}{2} u_n^2 + v_n \]

\[ \rho_n^{(3)} = \frac{1}{3} u_n^3 + u_n(v_{n-1} + v_n) \]

\[ \rho_n^{(4)} = \frac{1}{4} u_n^4 + u_n^2(v_{n-1} + v_n) + u_n u_{n+1} v_n + \frac{1}{2} v_n^2 + v_n v_{n+1} \]

\[ \rho_n^{(5)} = \frac{1}{5} u_n^5 + u_n^3(v_{n-1} + v_n) + u_n u_{n+1} v_n(u_n + u_{n+1}) + u_n v_{n-1}(v_{n-2} + v_{n-1} + v_n) + u_n v_n(v_{n-1} + v_n + v_{n+1}). \]
**GENERALIZED SYMMETRIES**

A vector function $G(..., u_{n-1}, u_n, u_{n+1}, ...) $ is a *symmetry* if the infinitesimal transformation $u_n \to u_n + \epsilon G(..., u_{n-1}, u_n, u_{n+1}, ...) $ leaves the lattice system invariant within order $\epsilon$. Consequently, $G$ must satisfy the linearized equation

$$ D_t G = F'(u_n)[G], $$

where $F'$ is the Fréchet derivative of $F$, i.e.,

$$ F'(u_n)[G] = \frac{\partial}{\partial \epsilon} F(u_n + \epsilon G)|_{\epsilon = 0}. $$

Here, $u_n \to u_n + \epsilon G(..., u_{n-1}, u_n, u_{n+1}, ...) $ means that $u_{n+k}$ is replaced by $u_{n+k} + \epsilon G|_{n \to n+k}.$

**Example**

Consider the Toda lattice

$$ \dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}). $$

Higher-order symmetry of rank $(3, 4)$:

$$ G_1 = v_n(u_n + u_{n+1}) - v_{n-1}(u_{n-1} + u_n), $$
$$ G_2 = v_n(u_{n+1}^2 - u_n^2) + v_n(v_{n+1} - v_{n-1}). $$
Algorithm for Generalized Symmetries of DDEs.

Consider the Toda system with \( w(u_n) = 1 \) and \( w(v_n) = 2 \). Compute the form of the symmetry of ranks \((3, 4)\), i.e. the first component of the symmetry has rank 3, the second rank 4.

**Step 1: Construct the form of the symmetry.**

List all monomials in \( u_n \) and \( v_n \) of rank 3 or less:

\[
\mathcal{L}_1 = \{ u_n^3, u_n^2, u_nv_n, u_n, v_n \},
\]

and of rank 4 or less:

\[
\mathcal{L}_2 = \{ u_n^4, u_n^3, u_n^2v_n, u_n^2, u_nv_n, u_n, v_n^2, v_n \}.
\]

For each monomial in \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \), introduce enough \( t \)-derivatives, so that each term exactly has rank 3 and 4, respectively.

Using the lattice equations, for the monomials in \( \mathcal{L}_1 \):

\[
\begin{align*}
\frac{d^0}{dt^0}(u_n^3) &= u_n^3, & \frac{d^0}{dt^0}(u_nv_n) &= u_nv_n, \\
\frac{d}{dt}(u_n^2) &= 2u_n\dot{u}_n = 2u_nv_{n-1} - 2u_nv_n, \\
\frac{d}{dt}(v_n) &= \dot{v}_n = u_nv_n - u_{n+1}v_n, \\
\frac{d^2}{dt^2}(u_n) &= \frac{d}{dt}(\dot{u}_n) = \frac{d}{dt}(v_{n-1} - v_n) \\
&= u_{n-1}v_{n-1} - u_nv_{n-1} - u_nv_n + u_{n+1}v_n.
\end{align*}
\]

Gather the resulting terms:

\[
\mathcal{R}_1 = \{ u_n^3, u_{n-1}v_{n-1}, u_nv_{n-1}, u_nv_n, u_{n+1}v_n \}.
\]
\[ R_2 = \{ u_n^4, u_{n-1}^2 v_{n-1}, u_{n-1} u_n v_{n-1}, u_n^2 v_{n-1}, v_{n-2} v_{n-1}, v_{n-1}^2, u_n^2 v_n, \\
  u_n u_{n+1} v_n, u_{n+1}^2 v_n, v_{n-1} v_n, v_n^2, v_n v_{n+1} \}. \]

Linearly combine the monomials in \( R_1 \) and \( R_2 \)

\[
G_1 = c_1 u_n^3 + c_2 u_{n-1} v_{n-1} + c_3 u_n v_{n-1} + c_4 u_n v_n + c_5 u_{n+1} v_n, \\
G_2 = c_6 u_n^4 + c_7 u_{n-1}^2 v_{n-1} + c_8 u_{n-1} u_n v_{n-1} + c_9 u_n^2 v_{n-1} \\
+ c_{10} v_n v_{n-1} + c_{11} v_n^2 v_{n-1} + c_{12} u_n^2 v_n + c_{13} u_n u_{n+1} v_n \\
+ c_{14} u_{n+1}^2 v_n + c_{15} v_{n-1} v_n + c_{16} v_n^2 + c_{17} v_n v_{n+1}. 
\]

**Step 2: Determine the unknown coefficients.**

Require that the symmetry condition holds.

Solution:

\[
c_1 = c_6 = c_7 = c_8 = c_9 = c_{10} = c_{11} = c_{13} = c_{16} = 0, \\
- c_2 = - c_3 = c_4 = c_5 = - c_{12} = c_{14} = - c_{15} = c_{17}. 
\]

Therefore, with \( c_{17} = 1 \), the symmetry of rank \((3, 4)\) is:

\[
G_1 = u_n v_n - u_{n-1} v_{n-1} + u_{n+1} v_n - u_n v_{n-1}, \\
G_2 = u_{n+1}^2 v_n - u_n^2 v_n + v_n v_{n+1} - v_n v_{n-1}. 
\]

Analogously, the symmetry of rank \((4, 5)\) reads

\[
G_1 = u_n^2 v_n + u_n u_{n+1} v_n + u_{n+1}^2 v_n + v_n v_{n+1} - u_{n-1}^2 v_{n-1} \\
- u_{n-1} u_n v_{n-1} - u_n^2 v_{n-1} - v_{n-2} v_{n-1} - v_{n-1}^2, \\
G_2 = u_{n+1} v_n^2 + 2 u_{n+1} v_n v_{n+1} + u_n v_n v_{n+1} - u_n^2 v_n + u_{n+1}^3 v_n \\
- u_{n-1} v_n v_{n-1} - 2 u_{n-1} v_n v_n - u_n v_n^2. 
\]
**Example: Nonlinear Schrödinger (NLS) equation.**

Ablowitz and Ladik discretization of the NLS equation:

\[ i \dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + u^*_n u_n(u_{n+1} + u_{n-1}). \]

\( u^*_n \) is the complex conjugate of \( u_n \).

Treat \( u_n \) and \( v_n = u^*_n \) as independent variables and add the complex conjugate equation. Absorb \( i \) in the scale on \( t \):

\[ \dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + u_n v_n(u_{n+1} + u_{n-1}), \]
\[ \dot{v}_n = -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n(v_{n+1} + v_{n-1}). \]

Since \( v_n = u^*_n \), \( w(v_n) = w(u_n) \).

No uniformity in rank! Introduce an auxiliary parameter \( \alpha \) with weight.

\[ \dot{u}_n = \alpha(u_{n+1} - 2u_n + u_{n-1}) + u_n v_n(u_{n+1} + u_{n-1}), \]
\[ \dot{v}_n = -\alpha(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n(v_{n+1} + v_{n-1}). \]

Uniformity in rank leads to

\[ w(u_n) + 1 = w(\alpha) + w(u_n) = 2w(u_n) + w(v_n) = 3w(u_n), \]
\[ w(v_n) + 1 = w(\alpha) + w(v_n) = 2w(v_n) + w(u_n) = 3w(v_n). \]

which yields

\[ w(u_n) = w(v_n) = \frac{1}{2}, w(\alpha) = 1. \]

Uniformity in rank is essential for steps 1 and 2.

After Step 2, set \( \alpha = 1 \). Step 3 leads to the result:

\[ \rho_n^{(1)} = c_1 u_n v_{n-1} + c_2 u_n v_{n+1}, \text{ etc.} \]
PART III: Software

• Scope and Limitations of Algorithms.

-- Systems of evolution equations or lattice equations must be polynomial in dependent variables. No explicitly dependencies on the independent variables.

-- Only one space variable (continuous or discretized) is allowed.

-- Program only computes polynomial conservation laws and generalized symmetries (no recursion operators yet).

-- Program computes conservation laws and symmetries that explicitly depend on the independent variables, if the highest degree is specified.

-- No limit on the number of equations in the system. In practice: time and memory constraints.

-- Input systems may have (nonzero) parameters. Program computes the compatibility conditions for parameters such that conservation laws and symmetries (of a given rank) exist.

-- Systems can also have parameters with (unknown) weight. This allows one to test evolution and lattice equations of non-uniform rank.

-- For systems where one or more of the weights is free, the program prompts the user for info.

-- Fractional weights and ranks are permitted.

-- Complex dependent variables are allowed.

-- PDEs and lattice equations must be of first-order in $t$. 
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<td>Conservation Laws</td>
<td>T. Wolf et al. School of Math. Sci. Queen Mary &amp; Westfield College University of London London E1 4NS, U.K.</td>
<td><a href="mailto:T.Wolf@maths.qmw.ac.uk">T.Wolf@maths.qmw.ac.uk</a></td>
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<td>DELiA (Pascal)</td>
<td>Conservation Laws and Generalized Symmetries</td>
<td>A. Bocharov et al. Saltire Software P.O. Box 1565 Beaverton, OR 97075 U.S.A.</td>
<td><a href="mailto:alexeib@saltire.com">alexeib@saltire.com</a></td>
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<td>V. Gerdt &amp; A. Zharkov Laboratory of Computing Techniques &amp; Automation Joint Institute for Nuclear Research 141980 Dubna, Russia</td>
<td><a href="mailto:gerdt@jinr.dubna.su">gerdt@jinr.dubna.su</a></td>
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<td>Invariants Symmetries.m (Mathematica)</td>
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<td>Ü. Göktaş &amp; W. Hereman Dept. of Math. Comp. Sci. Colorado School of Mines Golden, CO 80401, U.S.A.</td>
<td><a href="mailto:unalg@wolfram.com">unalg@wolfram.com</a> <a href="mailto:whereman@mines.edu">whereman@mines.edu</a></td>
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<td>SYMCD</td>
<td>Conservation Laws and Generalized Symmetries</td>
<td>M. Ito Dept. of Appl. Maths. Hiroshima University Higashi-Hiroshima 724 Japan</td>
<td><a href="mailto:ito@puramis.amath.hiroshima-u.ac.jp">ito@puramis.amath.hiroshima-u.ac.jp</a></td>
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<td>Generalized Symmetries</td>
<td>B. Fuchssteiner et al. Dept. of Mathematics Univ. of Paderborn D-33098 Paderborn Germany</td>
<td><a href="mailto:benno@uni-paderborn.de">benno@uni-paderborn.de</a></td>
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• Conclusions and Future Research

– Implement the recursion operator algorithm for PDEs.
– Design an algorithm for recursion operators of DDEs.
– Improve software, compare with other packages.
– Add tools for parameter analysis (Gröbner basis).
– Generalization towards broader classes of equations (e.g. \( u_{xt} \)).
– Generalization towards more space variables (e.g. KP equation).
– Conservation laws with time and space dependent coefficients.
– Conservation laws with \( n \) dependent coefficients.
– Exploit other symmetries in the hope to find conserved densities of non-polynomial form
– Application: test models for integrability.
– Application: study of classes of nonlinear PDEs or DDEs.
– Compute constants of motion for dynamical systems (e.g. Lorenz and Hénon-Heiles systems)
• Implementation in Mathematica – Software

– Ü. Göktaş and W. Hereman, The software package
  \textit{InvariantsSymmetries.m} and the related files are available at
  \textit{MathSource} is an electronic library of \textit{Mathematica} material.

– Software: available via FTP, ftp site \texttt{mines.edu} in
  \begin{verbatim}
  pub/papers/math_cs_dept/software/condens
  pub/papers/math_cs_dept/software/diffdens
  \end{verbatim}
  or via the Internet

  \texttt{URL: http://www.mines.edu/fs_home/whereman/}
• Publications


