A Symbolic Algorithm to Compute Conservation Laws of Nonlinear Evolution Equations

Willy Hereman

Department of Mathematical and Computer Sciences
Colorado School of Mines
Golden, CO 80401-1887

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OUTLINE

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• **Purpose**

  Design and implement algorithms to compute polynomial conservation laws, symmetries, and recursion operators for nonlinear systems of evolution and lattice equations.

• **Motivation**

  – Conservation laws describe the conservation of fundamental physical quantities (linear momentum, energy, etc.). Compare with constants of motion in mechanics.

  – Conservation laws provide a method to study quantitative and qualitative properties of equations and their solutions, e.g. Hamiltonian structures.

  – Conservation laws can be used to test numerical integrators.

  – For nonlinear PDEs and lattices, the existence of a sufficiently large (in principal infinite) number of conservation laws or symmetries assures complete **integrability**.
Conservation Laws of Evolution Equations (PDEs)

• System of evolution equations

\[ u_t = F(u, u_x, u_{2x}, \ldots, u_{mx}) \]

in a (single) space variable \( x \) and time \( t \), and with

\[ u = (u_1, u_2, \ldots, u_n), \quad F = (F_1, F_2, \ldots, F_n). \]

Notation:

\[ u_{mx} = u^{(m)} = \frac{\partial u}{\partial x^m}. \]

\( F \) is polynomial in \( u, u_x, \ldots, u_{mx} \).

PDEs of higher order in \( t \) should be recast as a first-order system.

• Examples:

The Korteweg-de Vries (KdV) equation:

\[ u_t + uu_x + u_{3x} = 0. \]

Fifth-order evolution equations with constant parameters \((\alpha, \beta, \gamma)\):

\[ u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma uu_3 x + u_{5x} = 0. \]

Special case. The fifth-order Sawada-Kotera (SK) equation:

\[ u_t + 5u^2 u_x + 5u_x u_{2x} + 5uu_3 x + u_{5x} = 0. \]

The Boussinesq (wave) equation:

\[ u_{tt} - u_{2x} + 3uu_{2x} + 3u_x^2 + \alpha u_{4x} = 0, \]

written as a first-order system (\( v \) auxiliary variable):

\[ u_t + v_x = 0, \]

\[ v_t + u_x - 3uu_x - \alpha u_{3x} = 0. \]
A vector nonlinear Schrödinger equation:

$$B_t + (|B|^2 B)_x + (B_0 \cdot B_x)B_0 + e \times B_{xx} = 0,$$

written in component form, $B_0 = (a, b)$ and $B = (u, v)$:

$$u_t + [u(u^2 + v^2) + \beta u + \gamma v - v_x]_x = 0,$$
$$v_t + [v(u^2 + v^2) + \theta u + \delta v + u_x]_x = 0,$$

$\beta = a^2, \gamma = \theta = ab, \text{ and } \delta = b^2.$

• **Conservation Laws.**

$$D_t \rho + D_x J = 0,$$

with conserved density $\rho$ and flux $J$.

Both are polynomial in $u, u_x, u_{2x}, u_{3x}, \ldots$.

$$P = \int_{-\infty}^{+\infty} \rho \, dx = \text{constant}$$

if $J$ vanishes at infinity.

Conserved densities are equivalent if they differ by a $D_x$ term.

**Example:** The Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{3x} = 0.$$

Conserved densities:

$$\rho_1 = u, \quad D_t(u) + D_x \left(\frac{u^2}{2} + u_{2x} \right) = 0.$$

$$\rho_2 = u^2, \quad D_t(u^2) + D_x \left(\frac{2u^3}{3} + 2uu_{2x} - u_x^2 \right) = 0.$$
\[ \rho_3 = u^3 - 3u_x^2, \]
\[ D_t (u^3 - 3u_x^2) + D_x \left( \frac{3}{4}u^4 - 6uu_x^2 + 3u^2u_{2x} + 3u_{2x}^2 - 6u_xu_{3x} \right) = 0. \]
\[ : \]
\[ \rho_6 = u^6 - 60u^3u_x^2 - 30u_x^4 + 108u^2u_{2x}^2 \]
\[ \quad + \frac{720}{7}u_{2x}^3 - \frac{64}{7}uu_{3x}^2 + \frac{216}{7}u_{4x}^2. \]

Time and space dependent conservation law:
\[ D_t (tu^2 - 2xu) \]
\[ + D_x \left( \frac{2}{3}tu^3 - xu^2 + 2tuu_{2x} - tu_x^2 - 2xu_{2x} + 2u_x \right) = 0. \]

- **Key concept:** Dilation invariance.

Conservation laws, symmetries and recursion operators are invariant under the dilation (scaling) symmetry of the given PDE.

The KdV equation, \( u_t + uu_x + u_{3x} = 0 \), has scaling symmetry
\[ (t, x, u) \rightarrow (\lambda^{-3}t, \lambda^{-1}x, \lambda^2u). \]

\( u \) corresponds to two \( x \)-derivatives, \( u \sim D_x^2 \). Similarly, \( D_t \sim D_x^3 \).

The weight, \( w \), of a variable equals the number of \( x \)-derivatives the variable carries.

Weights are rational. Weights of dependent variables are nonnegative.

Set \( w(D_x) = 1 \).
Due to dilation invariance: \( w(u) = 2 \) and \( w(D_t) = 3 \).
Consequently, \( w(x) = -1 \) and \( w(t) = -3 \).

The *rank* of a monomial is its total weight in terms of \( x \)-derivatives.

Every monomial in the KdV equation has rank 5.
The KdV equation is *uniform in rank*.

What do we do if equations are not uniform in rank?
Extend the space of dependent variables with parameters carrying weight.

Example: the Boussinesq system
\[
\begin{align*}
    u_t + v_x &= 0, \\
    v_t + u_x - 3uu_x - \alpha u_{3x} &= 0,
\end{align*}
\]
is not scaling invariant (\( u_x \) and \( u_{3x} \) are conflict terms).

Introduce an auxiliary parameter \( \beta \)
\[
\begin{align*}
    u_t + v_x &= 0, \\
    v_t + \beta u_x - 3uu_x - \alpha u_{3x} &= 0,
\end{align*}
\]
which has scaling symmetry:
\[
(x, t, u, v, \beta) \rightarrow (\lambda x, \lambda^2 t, \lambda^{-2} u, \lambda^{-3} v, \lambda^{-2} \beta).
\]

- **Algorithm for Conservation Laws of PDEs.**
  1. Determine weights (scaling properties) of variables and auxiliary parameters.
  2. Construct the form of the density (find monomial building blocks).
  3. Determine the constant coefficients.
• **Example:** Density of rank 6 for the KdV equation.

**Step 1: Compute the weights.**

Require uniformity in rank. With \( w(D_x) = 1 \):

\[
w(u) + w(D_t) = 2w(u) + 1 = w(u) + 3.
\]

Solve the linear system: \( w(u) = 2 \), \( w(D_t) = 3 \).

**Step 2: Determine the form of the density.**

List all possible powers of \( u \), up to rank 6: \([u, u^2, u^3]\).

Introduce \( x \) derivatives to ‘complete’ the rank.

- \( u \) has weight 2, introduce \( D_x^4 \).
- \( u^2 \) has weight 4, introduce \( D_x^2 \).
- \( u^3 \) has weight 6, no derivative needed.

Apply the \( D_x \) derivatives.

Remove terms of the form \( D_x u_{px} \), or \( D_x \) up to terms kept prior in the list.

\[
[u_{4x}] \rightarrow [\ ] \quad \text{empty list.}
\]
\[
[u_x^2, uu_{2x}] \rightarrow [u_x^2] \quad \text{since} \ uu_{2x} = (uu_x)_x - u_x^2.
\]
\[
[u^3] \rightarrow [u^3].
\]

Linearly combine the ‘building blocks’:

\[
\rho = c_1 u^3 + c_2 u_x^2.
\]
**Step 3: Determine the coefficients** $c_i$.

Compute $D_t \rho = 3c_1 u^2 u_t + 2c_2 u_x u_{xt}$.

Replace $u_t$ by $-(uu_x + u_3)$ and $u_{xt}$ by $-(uu_x + u_3)_x$.

Integrate the result, $E$, with respect to $x$. To avoid integration by parts, apply the Euler operator (variational derivative)

$$L_u = \sum_{k=0}^{m} (-D_x)^k \frac{\partial}{\partial u_{kx}}$$

$$= \frac{\partial}{\partial u} - D_x(\frac{\partial}{\partial u}) + D_x^2(\frac{\partial}{\partial u_{2x}}) + \cdots + (-1)^m D_x^m(\frac{\partial}{\partial u_{mx}}).$$

to $E$ of order $m$.

If $L_u(E) = 0$ immediately, then $E$ is a total $x$-derivative.

If $L_u(E) \neq 0$, the remaining expression must vanish identically.

$$D_t \rho = -D_x[\frac{3}{4} c_1 u^4 - (3c_1 - c_2) uu_x^2 + 3c_1 u^2 u_{2x}$$

$$- c_2 (u_{2x})^2 + 2c_2 u_x u_{3x}] - (3c_1 + c_2) u_x^3.$$  

The non-integrable term must vanish.

So, $c_1 = -\frac{1}{3} c_2$. Set $c_2 = -3$, hence, $c_1 = 1$.

Result:

$$\rho = u^3 - 3ux^2.$$  

Expression $\ldots$ yields

$$J = \frac{3}{4} u^4 - 6uu_x^2 + 3u^2 u_{2x} + 3u_{2x}^2 - 6u_x u_{3x}.$$  

Example: First few densities for the Boussinesq system:

$$\begin{align*}
\rho_1 &= u, & \rho_2 &= v, \\
\rho_3 &= uv, & \rho_4 &= \beta u^2 - u^3 + v^2 + \alpha u_x^2.
\end{align*}$$

(then substitute $\beta = 1$).
• Application.

A Class of Fifth-Order Evolution Equations

\[ u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0 \]

where \( \alpha, \beta, \gamma \) are nonzero parameters.

\[ u \sim D_x^2. \]

Special cases:

- \( \alpha = 30 \quad \beta = 20 \quad \gamma = 10 \) Lax.
- \( \alpha = 5 \quad \beta = 5 \quad \gamma = 5 \) Sawada – Kotera.
- \( \alpha = 20 \quad \beta = 25 \quad \gamma = 10 \) Kaup–Kupershmidt.
- \( \alpha = 2 \quad \beta = 6 \quad \gamma = 3 \) Ito.

What are the conditions for the parameters \( \alpha, \beta \) and \( \gamma \) so that the equation admits a density of fixed rank?

– Rank 2:
  
  No condition

  \[ \rho = u. \]

– Rank 4:
  
  Condition: \( \beta = 2\gamma \) (Lax and Ito cases)

  \[ \rho = u^2. \]
\begin{itemize}
  \item **Rank 6:**
    \begin{align*}
      \text{Condition:}
      10\alpha &= -2\beta^2 + 7\beta\gamma - 3\gamma^2 \\
      (\text{Lax, SK, and KK cases})
    \end{align*}
    \begin{align*}
      \rho &= u^3 + \frac{15}{(-2\beta + \gamma)}u_x^2.
    \end{align*}
  \item **Rank 8:**
    \begin{enumerate}
    \item $\beta = 2\gamma$ (Lax and Ito cases)
      \begin{align*}
        \rho &= u^4 - \frac{6\gamma}{\alpha}uu_x^2 + \frac{6}{\alpha}u_{2x}^2.
      \end{align*}
    \item $\alpha = -\frac{2\beta^2 - 7\beta\gamma - 4\gamma^2}{45}$ (SK, KK and Ito cases)
      \begin{align*}
        \rho &= u^4 - \frac{135}{2\beta + \gamma}uu_x^2 + \frac{675}{(2\beta + \gamma)^2}u_{2x}^2.
      \end{align*}
    \end{enumerate}
  \item **Rank 10:**
    \begin{align*}
      \text{Condition:}
      \beta &= 2\gamma
      \quad \text{and}
      10\alpha &= 3\gamma^2
      \quad (\text{Lax case})
    \end{align*}
    \begin{align*}
      \rho &= u^5 - \frac{50}{\gamma}u^2u_x^2 + \frac{100}{\gamma^2}uu_{2x}^2 - \frac{500}{7\gamma^3}u_{3x}^2.
    \end{align*}
\end{itemize}
What are the necessary conditions for the parameters $\alpha, \beta$ and $\gamma$ so that the equation admits $\infty$ many polynomial conservation laws?

- If $\alpha = \frac{3}{10} \gamma^2$ and $\beta = 2\gamma$ then there is a sequence (without gaps!) of conserved densities (Lax case).

- If $\alpha = \frac{1}{5} \gamma^2$ and $\beta = \gamma$ then there is a sequence (with gaps!) of conserved densities (SK case).

- If $\alpha = \frac{1}{5} \gamma^2$ and $\beta = \frac{5}{2} \gamma$ then there is a sequence (with gaps!) of conserved densities (KK case).

- If

$$\alpha = -\frac{2\beta^2 - 7\beta \gamma + 4\gamma^2}{45}$$

or

$$\beta = 2\gamma$$

then there is a conserved density of rank 8.

Combine both conditions: $\alpha = \frac{2\gamma^2}{9}$ and $\beta = 2\gamma$ (Ito case).
Conservation Laws of
Differential-difference (lattice) Equations

• Systems of lattices equations

Consider the system of lattice equations, continuous in time, discretized in (one dimensional) space

\[ \dot{u}_n = F(..., u_{n-1}, u_n, u_{n+1}, ...) \]

where \( u_n \) and \( F \) are vector dynamical variables.

\( F \) is polynomial with constant coefficients.

No restrictions on the level of the shifts or the degree of nonlinearity.

• Conservation Laws.

\[ \dot{\rho}_n = J_n - J_{n+1} \]

with density \( \rho_n \) and flux \( J_n \).

Both are polynomials in \( u_n \) and its shifts.

\[ \frac{d}{dt} (\sum_n \rho_n) = \sum_n \dot{\rho}_n = \sum_n (J_n - J_{n+1}) \]

if \( J_n \) is bounded for all \( n \).

Subject to suitable boundary or periodicity conditions

\[ \sum_n \rho_n = \text{constant}. \]
• Example.

Consider the one-dimensional Toda lattice
\[ \ddot{y}_n = \exp(y_{n-1} - y_n) - \exp(y_n - y_{n+1}) \]
y
is the displacement from equilibrium of the \( n \)th particle with unit mass under an exponential decaying interaction force between nearest neighbors.

Change of variables:
\[ u_n = \dot{y}_n, \quad v_n = \exp(y_n - y_{n+1}) \]
yields
\[ \ddot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}). \]

Toda system is completely integrable.

The first two density-flux pairs (computed by hand):
\[ \rho_n^{(1)} = u_n, \quad J_n^{(1)} = v_{n-1}, \quad \text{and} \quad \rho_n^{(2)} = \frac{1}{2}u_n^2 + v_n, \quad J_n^{(2)} = u_nv_{n-1}. \]

• Key concept: Dilation invariance.

The Toda system as well as the conservation laws and symmetries are invariant under the dilation symmetry
\[ (t, u_n, v_n) \rightarrow (\lambda^{-1}t, \lambda u_n, \lambda^2v_n). \]

• Example: Nonlinear Schrödinger (NLS) equation.

Ablowitz and Ladik discretization of the NLS equation:
\[ i\dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + u_n^*u_n(u_{n+1} + u_{n-1}). \]
u
is the complex conjugate of \( u_n \).
Treat \( u_n \) and \( v_n = u_n^* \) as independent variables and add the complex conjugate equation. Absorb \( i \) in the scale on \( t \):

\[
\begin{align*}
\dot{u}_n &= u_{n+1} - 2u_n + u_{n-1} + u_n v_n (u_{n+1} + u_{n-1}), \\
\dot{v}_n &= -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}).
\end{align*}
\]

Since \( v_n = u_n^* \), \( w(v_n) = w(u_n) \).

No uniformity in rank! Introduce an auxiliary parameter \( \alpha \) with weight.

\[
\begin{align*}
\dot{u}_n &= \alpha (u_{n+1} - 2u_n + u_{n-1}) + u_n v_n (u_{n+1} + u_{n-1}), \\
\dot{v}_n &= -\alpha (v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}).
\end{align*}
\]

Uniformity in rank leads to

\[
\begin{align*}
w(u_n) + 1 &= w(\alpha) + w(u_n) = 2w(u_n) + w(v_n) = 3w(u_n), \\
w(v_n) + 1 &= w(\alpha) + w(v_n) = 2w(v_n) + w(u_n) = 3w(v_n).
\end{align*}
\]

which yields

\[ w(u_n) = w(v_n) = \frac{1}{2}, w(\alpha) = 1. \]

Uniformity in rank is essential for steps 1 and 2. After Step 2, set \( \alpha = 1 \). Step 3 leads to the result:

\[
\rho_n^{(1)} = c_1 u_n v_{n-1} + c_2 u_n v_{n+1}, \quad \text{etc.}
\]
Software

• Scope and Limitations of Algorithms.

– Systems of evolution equations or lattice equations must be polynomial in dependent variables. No explicitly dependencies on the independent variables.
– Only one space variable (continuous or discretized) is allowed.
– Program only computes polynomial conservation laws and generalized symmetries (no recursion operators yet).
– Program computes conservation laws and symmetries that explicitly depend on the independent variables, if the highest degree is specified.
– No limit on the number of equations in the system. In practice: time and memory constraints.
– Input systems may have (nonzero) parameters. Program computes the compatibility conditions for parameters such that conservation laws and symmetries (of a given rank) exist.
– Systems can also have parameters with (unknown) weight. This allows one to test evolution and lattice equations of non-uniform rank.
– For systems where one or more of the weights is free, the program prompts the user for info.
– Fractional weights and ranks are permitted.
– Complex dependent variables are allowed.
– PDEs and lattice equations must be of first-order in $t$. 
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<td>CONLAW 1/2/3 (REDUCE)</td>
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<td>T. Wolf et al. School of Math. Sci. Queen Mary &amp; Westfield College University of London London E1 4NS, U.K.</td>
<td><a href="mailto:T.Wolf@maths.qmw.ac.uk">T.Wolf@maths.qmw.ac.uk</a></td>
</tr>
<tr>
<td>DELiA (Pascal)</td>
<td>Conservation</td>
<td>A. Bocharov et al. Saltire Software P.O. Box 1565 Beaverton, OR 97075 U.S.A.</td>
<td><a href="mailto:alexeib@saltire.com">alexeib@saltire.com</a></td>
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<tr>
<td>FS (REDUCE)</td>
<td>Conservation</td>
<td>V. Gerdt &amp; A. Zharkov Laboratory of Computing Techniques &amp; Automation Joint Institute for Nuclear Research 141980 Dubna, Russia</td>
<td><a href="mailto:gerdt@jinr.dubna.su">gerdt@jinr.dubna.su</a></td>
</tr>
<tr>
<td>Invariants Symmetries.m (Mathematica)</td>
<td>Conservation</td>
<td>Ü. Göktaş &amp; W. Hereman Dept. of Math. Comp. Sci. Colorado School of Mines Golden, CO 80401, U.S.A.</td>
<td><a href="mailto:unalg@wolfram.com">unalg@wolfram.com</a> <a href="mailto:whereman@mines.edu">whereman@mines.edu</a></td>
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<td>SYMCD (REDUCE)</td>
<td>Conservation Laws and Generalized Symmetries</td>
<td>M. Ito Dept. of Appl. Maths. Hiroshima University Higashi-Hiroshima 724 Japan</td>
<td><a href="mailto:ito@puramis.amath.hiroshima-u.ac.jp">ito@puramis.amath.hiroshima-u.ac.jp</a></td>
</tr>
<tr>
<td>symmetry &amp; mastersymmetry (MuPAD)</td>
<td>Generalized Symmetries</td>
<td>B. Fuchssteiner et al. Dept. of Mathematics Univ. of Paderborn D-33098 Paderborn Germany</td>
<td><a href="mailto:benno@uni-paderborn.de">benno@uni-paderborn.de</a></td>
</tr>
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Conclusions and Future Research

- Improve software, compare with other packages.
- Add tools for parameter analysis (Gröbner basis).
- Generalization towards broader classes of equations (e.g. $u_{xt}$).
- Generalization towards more space variables (e.g. KP equation).
- Conservation laws with time and space dependent coefficients.
- Conservation laws with $n$ dependent coefficients.
- Exploit other symmetries in the hope to find conserved densities. of non-polynomial form
- Application: test models for integrability.
- Application: study of classes of nonlinear PDEs or DDEs.
- Compute constants of motion for dynamical systems (e.g. Lorenz and Hénon-Heiles systems)
• Implementation in Mathematica – Software


– Software: available via the Internet

URL: http://www.mines.edu/fs_home/whereman/
• Publications


