Symbolic Computation of Travelling Wave Solutions of Nonlinear PDEs and Lattices with Mathematica

Prof. Willy Hereman
Department of Mathematical and Computer Sciences
Colorado School of Mines
Golden, CO-80401, U.S.A.
http://www.mines.edu/fs_home/whereman/
whereman@mines.edu

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Collaborators: Ünal Göktaş (Wolfram Research, Inc.)
Ryan Martino, Joel Miller, Linda Hong (REU ’99)
Doug Baldwin, Steve Formaneck, Andrew Menz (REU ’00)
Doug Baldwin, Ben Kowalski (REU ’01)

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Purpose & Motivation

- **Develop** and implement various **methods** to find closed form solutions of nonlinear PDEs and DDEs: Lie symmetry methods, similarity methods, etc.

- **Fully automate** the hyperbolic and elliptic function methods to compute exact solitary wave solutions of nonlinear partial differential equations (PDEs) and differential-difference equations (DDEs or lattices).

- **Class** of nonlinear PDEs and DDEs solvable with such methods includes famous evolution and wave equations. Typical examples: Korteweg-de Vries, Fisher and Boussinesq PDEs, Toda and Volterra lattices (DDEs).

- Solutions of tanh (kink) or sech (pulse) type **model** solitary waves in fluid dynamics, plasmas, electrical circuits, optical fibers, biogenetics, etc.

- **Benchmark** solutions for numerical PDE solvers.

- **Research aspect**: Design high-quality application packages to compute solitary wave solutions of large classes of nonlinear evolution and wave equations.

- **Educational aspect**: Software as course ware for courses in nonlinear PDEs, theory of nonlinear waves, integrability, dynamical systems, and modeling with symbolic software. REU Projects.

- **Users**: scientists working on nonlinear wave phenomena in fluid dynamics, nonlinear networks, elastic media, chemical kinetics, material science, bio-sciences, plasma physics, and nonlinear optics.
Typical Examples of ODEs and PDEs

• The Duffing equation:

\[ u'' + u + \alpha u^3 = 0 \]

Solutions in terms of elliptic functions:

\[ u(x) = \pm \frac{\sqrt{c_1^2 - 1}}{\sqrt{\alpha}} \cn(c_1 x + \Delta; \frac{c_1^2 - 1}{2c_1^2}), \]

and

\[ u(x) = \pm \frac{\sqrt{2(c_1^2 - 1)}}{\sqrt{\alpha}} \sn(c_1 x + \Delta; \frac{1 - c_1^2}{c_1^2}). \]

• The Korteweg-de Vries (KdV) equation:

\[ u_t + 6\alpha uu_x + u_{3x} = 0. \]

Solitary wave solution:

\[ u(x, t) = \frac{8c_1^3 - c_2}{6\alpha c_1} - \frac{2c_1^2}{\alpha} \tanh^2 [c_1 x + c_2 t + \Delta], \]

or, equivalently,

\[ u(x, t) = -\frac{4c_1^3 + c_2}{6\alpha c_1} + \frac{2c_1^2}{\alpha} \sech^2 [c_1 x + c_2 t + \Delta]. \]

Cnoidal wave solution:

\[ u(x, t) = \frac{4c_1^3 (1 - 2m) - c_2}{\alpha c_1} + \frac{12m c_1^2}{\alpha} \cn^2 (c_1 x + c_2 t + \Delta; m), \]

modulus \( m \).
• The modified Korteweg-de Vries (mKdV) equation:
\[ u_t + \alpha u^2 u_x + u_{3x} = 0. \]
Solitary wave solution:
\[ u(x, t) = \pm \sqrt{\frac{6}{\alpha}} c_1 \text{sech} [c_1 x - c_1^3 t + \Delta]. \]

• Three-dimensional modified Korteweg-de Vries equation:
\[ u_t + 6 u^2 u_x + u_{xyz} = 0. \]
Solitary wave solution:
\[ u(x, y, z, t) = \pm \sqrt{c_2 c_3} \text{sech} [c_1 x + c_2 y + c_3 z - c_1 c_2 c_3 t + \Delta]. \]

• The combined KdV-mKdV equation:
\[ u_t + 6 \alpha u u_x + 6 \beta u^2 u_x + \gamma u_{3x} = 0. \]
Real solitary wave solution:
\[ u(x, t) = -\frac{\alpha}{2\beta} \pm \frac{\gamma}{\beta} c_1 \text{sech}(c_1 x + \frac{c_1}{2\beta}(3\alpha^2 - 2\beta \gamma c_1^2) t + \Delta). \]
Complex solutions:
\[ u(x, t) = -\frac{\alpha}{2\beta} \pm i \frac{\gamma}{\beta} c_1 \text{tanh}(c_1 x + \frac{c_1}{2\beta}(3\alpha^2 + 4\beta \gamma c_1^2) t + \Delta), \]
\[ u(x, t) = -\frac{\alpha}{2\beta} + \frac{1}{2} \frac{\gamma}{\beta} c_1 \left( \text{sech} \xi \pm i \text{tanh} \xi \right), \]
and
\[ u(x, t) = -\frac{\alpha}{2\beta} - \frac{1}{2} \frac{\gamma}{\beta} c_1 \left( \text{sech} \xi \mp i \text{tanh} \xi \right), \]
with \( \xi = c_1 x + \frac{c_1}{2\beta}(3\alpha^2 + \beta \gamma c_1^2) t + \Delta. \)
- The Fisher equation:

\[ u_t - u_{xx} - u (1 - u) = 0. \]

Solitary wave solution:

\[ u(x, t) = \frac{1}{4} \pm \frac{1}{2} \tanh \xi + \frac{1}{4} \tanh^2 \xi, \]

with

\[ \xi = \pm \frac{1}{2\sqrt{6}} x \pm \frac{5}{12} t + \Delta. \]

- The generalized Kuramoto-Sivashinski equation:

\[ u_t + uu_x + u_{xx} + \sigma u_{3x} + u_{4x} = 0. \]

Solitary wave solutions

(ignoring symmetry \( u \to -u, x \to -x, \sigma \to -\sigma \)):

For \( \sigma = 4 \):

\[ u(x, t) = 9 - 2c_2 - 15 \tanh \xi \left( 1 + \tanh \xi - \tanh^2 \xi \right) \]

with \( \xi = \frac{x}{2} + c_2 t + \Delta. \)

For \( \sigma = \frac{12}{\sqrt{47}} \):

\[ u(x, t) = \frac{45 \mp 4418c_2}{47\sqrt{47}} \pm \frac{45}{47\sqrt{47}} \tanh \xi - \frac{45}{47\sqrt{47}} \tanh^2 \xi \pm \frac{15}{47\sqrt{47}} \tanh^3 \xi \]

with \( \xi = \pm \frac{1}{2\sqrt{47}} x + c_2 t + \Delta. \)
For $\sigma = 16/\sqrt{73}$:

$$u(x, t) = \frac{2 (30 + 5329c_2)}{73\sqrt{73}} \pm \frac{75}{73\sqrt{73}} \tanh \xi - \frac{60}{73\sqrt{73}} \tanh^2 \xi \pm \frac{15}{73\sqrt{73}} \tanh^3 \xi$$

with $\xi = \pm \frac{1}{2\sqrt{73}} x + c_2 t + \Delta$.

For $\sigma = 0$:

$$u(x, t) = -2 \left[ \frac{19}{11} c_2 - \frac{135}{11} \frac{\tanh \xi}{\sqrt{11}} + \frac{165}{19} \frac{\tanh^3 \xi}{\sqrt{19}} \right]$$

with $\xi = \frac{1}{2} \sqrt{\frac{11}{19}} x + c_2 t + \Delta$.

- The Boussinesq (wave) equation:

$$u_{tt} - u_{2x} + 3uu_{2x} + 3u_x^2 + \alpha u_{4x} = 0,$$

or written as a first-order system ($v$ auxiliary variable):

$$u_t + v_x = 0,$$
$$v_t + u_x - 3uu_x - \alpha u_{3x} = 0.$$

Solitary wave solution:

$$u(x, t) = \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2 [c_1 x + c_2 t + \Delta],$$
$$v(x, t) = b_0 + 4\alpha c_1 c_2 \tanh^2 [c_1 x + c_2 t + \Delta].$$

- The Broer-Kaup system:

$$u_{ty} + 2(uu_x)_y + 2v_{xx} - u_{xy} = 0,$$
$$v_t + 2(\nu v)_x + v_{xx} = 0.$$

Solitary wave solution:

$$u(x, t) = -\frac{c_3}{2c_1} + c_1 \tanh [c_1 x + c_2 y + c_3 t + \Delta],$$
$$v(x, t) = c_1 c_2 - c_1 c_2 \tanh^2 [c_1 x + c_2 y + c_3 t + \Delta].$$
System of three nonlinear coupled equations (Gao & Tian, 2001):

\[
\begin{align*}
    u_t - u_x - 2v &= 0, \\
    v_t + 2uw &= 0, \\
    w_t + 2uv &= 0.
\end{align*}
\]

Solutions:

\[
\begin{align*}
    u(x, t) &= \pm c_2 \tanh \xi, \\
    v(x, t) &= \pm \frac{1}{2} c_2 (c_1 - c_2) \sech^2 \xi, \\
    w(x, t) &= -\frac{1}{2} c_2 (c_1 - c_2) \sech^2 \xi,
\end{align*}
\]

and

\[
\begin{align*}
    u(x, t) &= \pm ic_2 \sech \xi, \\
    v(x, t) &= \pm \frac{1}{2} ic_2 (c_1 - c_2) \tanh \xi \sech \xi, \\
    w(x, t) &= \frac{1}{4} c_2 (c_1 - c_2) \left(1 - 2 \sech^2 \xi\right) ,
\end{align*}
\]

and also

\[
\begin{align*}
    u(x, t) &= \pm \frac{1}{2} ic_2 \left(\sech \xi + i \tanh \xi\right) , \\
    v(x, t) &= \pm \frac{1}{4} c_2 (c_1 - c_2) \sech \xi \left(\sech \xi + i \tanh \xi\right) , \\
    w(x, t) &= -\frac{1}{4} c_2 (c_1 - c_2) \sech \xi \left(\sech \xi + i \tanh \xi\right) 
\end{align*}
\]

with \( \xi = c_1 x + c_2 t + \Delta \).
• Nonlinear sine-Gordon equation (light cone coordinates):

\[ \Phi_{xt} = \sin \Phi. \]

Set \( u = \Phi_x, \ v = \cos(\Phi) - 1, \)

\[ u_{xt} - u - u v = 0, \]
\[ u_t^2 + 2v + v^2 = 0. \]

Solitary wave solution (kink):
\[ u = \pm \frac{1}{\sqrt{-c}} \text{sech}\left[ \frac{1}{\sqrt{-c}}(x - ct) + \Delta \right], \]
\[ v = 1 - 2 \text{sech}^2\left[ \frac{1}{\sqrt{-c}}(x - ct) + \Delta \right]. \]

Solution:
\[ \Phi(x, t) = \int u(x, t) dx = \pm 4 \arctan \left( \exp \left( \frac{1}{\sqrt{-c}}(x - ct) + \Delta \right) \right). \]

• ODEs from quantum field theory:

\[ u_{xx} = -u + u^3 + auv^2, \]
\[ v_{xx} = bv + cv^3 + av(u^2 - 1). \]

Solitary wave solutions:
\[ u = \pm \tanh\left[ \sqrt{\frac{a^2 - c}{2(a-c)}} x + \Delta \right], \]
\[ v = \pm \sqrt{\frac{1-a}{a-c}} \text{sech}\left[ \sqrt{\frac{a^2 - c}{2(a-c)}} x + \Delta \right], \]

provided \( b = \sqrt{\frac{a^2-c}{2(a-c)}}. \)
Typical Examples of DDEs (lattices)

• The Toda lattice:
  \[ \ddot{u}_n = (1 + \dot{u}_n)(u_{n-1} - 2u_n + u_{n+1}). \]
  Solitary wave solution:
  \[ u_n(t) = a_0 \pm \sinh(c_1) \tanh[c_1n \pm \sinh(c_1)t + \Delta]. \]

• The Volterra lattice:
  \[ \dot{u}_n = u_n(v_n - v_{n-1}), \quad \dot{v}_n = v_n(u_{n+1} - u_n). \]
  Solitary wave solution:
  \[ u_n(t) = -c_2 \coth(c_1) + c_2 \tanh [c_1n + c_2t + \Delta], \]
  \[ v_n(t) = -c_2 \coth(c_1) - c_2 \tanh [c_1n + c_2t + \Delta]. \]

• The Relativistic Toda lattice:
  \[ \dot{u}_n = (1 + \alpha u_n)(v_n - v_{n-1}), \quad \dot{v}_n = v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}). \]
  Solitary wave solution:
  \[ u_n(t) = -c_2 \coth(c_1) - \frac{1}{\alpha} + c_2 \tanh [c_1n + c_2t + \Delta], \]
  \[ v_n(t) = \frac{c_2 \coth(c_1)}{\alpha} - \frac{c_2}{\alpha} \tanh [c_1n + c_2t + \Delta]. \]
Algorithm for Tanh Solutions for system of PDEs

Given: System of nonlinear PDEs of order $m$

$$\Delta(u(x), u'(x), u''(x), \ldots, u^{(m)}(x)) = 0.$$ 

Dependent variable $u$ has $M$ components $u_i$ (or $u, v, w, \ldots$).
Independent variable $x$ has $N$ components $x_j$ (or $x, y, z, \ldots, t$).

**Step T1:**

- Seek solution $u(x) = U(T)$, with

$$T = \tanh \xi = \tanh \left[ \sum_j c_j x_j + \Delta \right].$$

- Observe $\tanh' \xi = 1 - \tanh^2 \xi$ or $T' = 1 - T^2$. Hence, all derivative of $T$ are polynomial in $T$. For example, $T'' = -2T(1 - T^2)$, etc.

- Repeatedly apply the operator rule

$$\frac{\partial \bullet}{\partial x_j} = \frac{d \bullet}{dT} \frac{\partial T}{\partial x_j} = c_j (1 - T^2) \frac{d \bullet}{dT}.$$ 

Produces a nonlinear system of ODEs

$$\Delta(T, U(T), U'(T), U''(T), \ldots, U^{(m)}(T)) = 0.$$ 

**NOTE:** Compare with the ultra-spherical (linear) ODE:

$$(1 - x^2)y''(x) - (2\alpha + 1)xy'(x) + n(n + 2\alpha)y(x) = 0$$

with integer $n \geq 0$ and $\alpha$ real. Includes:

* Legendre equation ($\alpha = \frac{1}{2}$),
* ODE for Chebyshev polynomials of type I ($\alpha = 0$),
* ODE for Chebyshev polynomials of type II ($\alpha = 1$).
Example: For the Boussinesq system
\[
\begin{aligned}
\frac{\partial u}{\partial t} + v_x &= 0, \\
\frac{\partial v}{\partial t} + u_x - 3uu_x - \alpha u_{3x} &= 0,
\end{aligned}
\]
after cancelling common factors \(1 - T^2\),
\[
\begin{aligned}
c_2 U' + c_1 V' &= 0, \\
c_2 V' + c_1 U' - 3c_1 UU' \\
+ \alpha c_1^3 \left[ 2(1 - 3T^2)U' + 6T(1 - T^2)U'' - (1 - T^2)^2 U'''ight] &= 0.
\end{aligned}
\]

**Step T2:**

- Seek polynomial solutions
  \[
  U_i(T) = \sum_{j=0}^{M_i} a_{ij} T^j.
  \]

Determine the highest exponents \(M_i \geq 1\).
Substitute \(U_i(T) = T^{M_i}\) into the LHS of ODE.
Gives polynomial \(P(T)\).
For every \(P_i\) consider all possible balances of the highest exponents in \(T\).
Solve the resulting linear system(s) for the unknowns \(M_i\).

- Example: Balance highest exponents for the Boussinesq system
  \[
  M_1 - 1 = M_2 - 1, \quad 2M_1 - 1 = M_1 + 1.
  \]
So, \(M_1 = M_2 = 2\).
Hence,
\[
\begin{aligned}
U(T) &= a_{10} + a_{11} T + a_{12} T^2, \\
V(T) &= a_{20} + a_{21} T + a_{22} T^2.
\end{aligned}
\]
Step T3:

- Derive algebraic system for the unknown coefficients \( a_{ij} \) by setting to zero the coefficients of the power terms in \( T \).
- Example: Algebraic system for Boussinesq case
  
  \[
  a_{11} c_1 (3a_{12} + 2\alpha c_1^2) = 0, \\
  a_{12} c_1 (a_{12} + 4\alpha c_1^2) = 0, \\
  a_{21} c_1 + a_{11} c_2 = 0, \\
  a_{22} c_1 + a_{12} c_2 = 0, \\
  a_{11} c_1 - 3a_{10} a_{11} c_1 + 2\alpha a_{11} c_1^3 + a_{21} c_2 = 0, \\
  -3a_{11}^2 c_1 + 2a_{12} c_1 - 6a_{10} a_{12} c_1 + 16\alpha a_{12} c_1^3 + 2a_{22} c_2 = 0.
  \]

Step T4:

- Solve the nonlinear algebraic system with parameters.
- Example: Solution for Boussinesq system
  
  \[
  a_{10} = \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2}, \quad a_{11} = 0, \\
  a_{12} = -4\alpha c_1^2, \quad a_{20} = \text{free}, \\
  a_{21} = 0, \quad a_{22} = 4\alpha c_1 c_2.
  \]

Step T5:

- Return to the original variables. Test the final solution(s) of PDE. Reject trivial solutions.
- Example: Solitary wave solution for Boussinesq system:
  
  \[
  u(x, t) = \frac{c_1^2 - c_2^2 + 8\alpha c_1^4}{3c_1^2} - 4\alpha c_1^2 \tanh^2 \left[ c_1 x + c_2 t + \Delta \right], \\
  v(x, t) = a_{20} + 4\alpha c_1 c_2 \tanh^2 \left[ c_1 x + c_2 t + \Delta \right].
  \]
Algorithm for Sech Solutions for system of PDEs

Given: System of PDEs of order \( m \)
\[
\Delta(u(x), u'(x), u''(x), \ldots, u^{(m)}(x)) = 0.
\]
Dependent variable \( u \) has \( M \) components \( u_i \) (or \( u, v, w, \ldots \)).
Independent variable \( x \) has \( N \) components \( x_j \) (or \( x, y, z, \ldots, t \)).

Step S1:

- Seek solution \( u_i(x) = U_i(S) \), with
  \[
  S = \text{sech}\xi = \text{sech}\left[ \sum_j c_j x_j + \Delta \right].
  \]
- Observe \( (\text{sech}\xi)' = -\tanh\xi \text{sech}\xi \) or \( S' = -TS = -\sqrt{1 - S^2}S \).
- Repeatedly apply the operator rule
  \[
  \frac{\partial\bullet}{\partial x_j} = \frac{d\bullet}{dS} \frac{\partial S}{\partial x_j} = -c_j S \sqrt{1 - S^2} \frac{d\bullet}{dS}.
  \]

Leads to coupled system of nonlinear ODEs
\[
\Gamma(S, U(S), U'(S), \ldots) + \sqrt{1 - S^2} \Pi(S, U(S), U'(S), \ldots) = 0.
\]

All components of \( \Gamma \) and \( \Pi \) are polynomial ODEs.

First case: \( \Gamma = 0 \) or \( \Pi = 0 \).
\[
\Delta(S, U(S), U'(S), \ldots) = 0.
\]
\( \Delta \) stands for either \( \Gamma \) or \( \Pi \).

Note: All terms in the given system of PDE must be of even or odd order.
Example: For the 3D mKdV equation
\[ u_t + 6u^2u_x + u_{xyz} = 0, \]
after cancelling a common factor \(-\sqrt{1 - S^2} S\),
\[ c_4 U' + 6c_1 U^2 U' + c_1 c_2 c_3 [(1 - 6S^2) U' + 3S(1 - 2S^2) U'' + S^2(1 - S^2) U'''] = 0. \]

**Step S2:**

- Seek polynomial solutions
  \[ U_i(S) = \sum_{j=0}^{M_i} a_{ij} S^j. \]

  Substitute \( U_i(S) = S^{M_i} \) and balance the highest power terms in \( S \) to determine \( M_i \).

- Example: Balance of exponents for the 3D mKdV case
  \[ 3M_1 - 1 = M_1 + 1. \]
  So, \( M_1 = 1 \). Hence,
  \[ U(S) = a_{10} + a_{11} S. \]

**Step S3:**

- Derive algebraic system for the unknown coefficients \( a_{ij} \) by setting to zero the coefficients of the power terms in \( S \).

- Example: Algebraic system for 3D mKdV case
  \[ a_{11} c_1 (a_{11}^2 - c_2 c_3) = 0, \]
  \[ a_{11} (6a_{10}^2 c_1 + c_1 c_2 c_3 + c_4) = 0, \]
  \[ a_{10} a_{11}^2 c_1 = 0. \]
Step S4:

- Solve the nonlinear algebraic system with parameters.
- Example: Solution for 3D mKdV case

\[
\begin{align*}
a_{10} &= 0, \\
a_{11} &= \pm \sqrt{c_1 c_3}, \\
c_4 &= -c_1 c_2 c_3.
\end{align*}
\]

Step S5:

- Return to the original variables. Test the final solution(s). Reject trivial solutions.
- Example: Solitary wave solution for the 3D mKdV equation

\[
u(x, y, z, t) = \pm \sqrt{c_2 c_3} \text{sech}(c_1 x + c_2 y + c_3 z - c_1 c_2 c_3 t).
\]

Second case: $\Gamma \neq 0$ and $\Pi \neq 0$.

\[
\Gamma(S, U(S), U'(S), \ldots) + \sqrt{1 - S^2} \Pi(S, U(S), U'(S), \ldots) = 0.
\]

Most general solution

\[
U_i(S) = \sum_{j=0}^{\hat{M}_i} \sum_{k=0}^{\hat{N}_i} \tilde{a}_{i,j,k} S^j T^k.
\]

Double series is not necessary! Solution can be rearranged as

\[
U_i(S) = \sum_{j=0}^{M_i} a_{ij} S^j + T \sum_{j=0}^{N_i} b_{ij} S^j.
\]
Algorithm for Mixed Tanh/Sech Solutions for PDEs

Step ST1:

- Seek solution in \( u_i(x) = U_i(S) \), with
  \[
  S = \text{sech}\xi = \text{sech}\left[\sum_{j} c_j x_j + \Delta\right].
  \]

Repeatedly apply the operator rule
\[
\frac{\partial \bullet}{\partial x_j} = \frac{d \bullet}{dS} \frac{\partial S}{\partial x_j} = -c_j S \sqrt{1 - S^2} \frac{d \bullet}{dS}.
\]

- Example: Coupled system due to Gao and Tian (2001)
  \[
  u_t - u_x - 2v = 0,
  \]
  \[
  v_t + 2uw = 0,
  \]
  \[
  w_t + 2uv = 0,
  \]

transforms into
\[
(c_1 - c_2) S \sqrt{1 - S^2} U' - 2V = 0,
\]
\[
c_2 S \sqrt{1 - S^2} V' - 2UW = 0,
\]
\[
c_2 S \sqrt{1 - S^2} W' - 2UV = 0.
\]

Step ST2:

- Seek solution
  \[
  U_i(S) = \sum_{j=0}^{M_i} a_{ij} S^j + \sqrt{1 - S^2} \sum_{j=0}^{N_i} b_{ij} S^j.
  \]

First, determine the leading exponents \( M_i, N_i \). Substitute
\[
U_i(S) = a_{i0} + a_i M_i S^{M_i} + \sqrt{1 - S^2} (b_{i0} + b_{iN_i} S^{N_i})
\]
to get
\[ P(S) + \sqrt{1 - S^2} Q(S) = 0. \]

\( P \) and \( Q \) are polynomials.

Consider possible balances of the highest exponents in \( P_i \) and \( Q_i \).

Get a linear system of \( 2M \) (or less) equations for the \( 2M \) unknown \( M_i \) and \( N_i \).

No longer assume \( M_i \geq 1, N_i \geq 1 \) (some \( M_i \) or \( N_i \) may be zero).

**Trouble.** Strongly underdetermined problem. Set all \( M_i = 2 \) and \( N_i = 1 \).

- **Example:** Quadratic solutions in \( S \) and \( T \) only.

Substitute
\[
\begin{align*}
U(S) &= a_{10} + a_{11}S + a_{12}S^2 + \sqrt{1 - S^2} (b_{10} + b_{11}S), \\
V(S) &= a_{20} + a_{21}S + a_{22}S^2 + \sqrt{1 - S^2} (b_{20} + b_{21}S), \\
W(S) &= a_{30} + a_{31}S + a_{32}S^2 + \sqrt{1 - S^2} (b_{30} + b_{31}S).
\end{align*}
\]

leads to
\[ P(S) + \sqrt{1 - S^2} Q(S) = 0, \]

\( P \) and \( Q \) are polynomials.

**Step ST3:**

- Derive the algebraic system for the coefficients \( a_{ij}, b_{ij} \) by setting to zero the coefficients of power terms in \( S \) in \( P = 0 \) and \( Q = 0 \) separately.

- **Example:** Algebraic system has 25 equations (not shown).
Step ST4:

- Solve the nonlinear algebraic system with parameters.
- Example: 11 solutions in total: 3 are trivial ($U_i = \text{constant}$), 8 are nontrivial.

Step ST5:

- Return to the original variables. Test the final solution(s). Reject trivial (constant) solutions.
- Example: Solitary wave solutions:

\[
\begin{align*}
  u(x,t) &= \pm c_2 \tanh \xi, \\
  v(x,t) &= \mp \frac{1}{2} c_2 (c_1 - c_2) \sech^2 \xi, \\
  w(x,t) &= -\frac{1}{2} c_2 (c_1 - c_2) \sech^2 \xi,
\end{align*}
\]

(could have been obtained with tanh-method), and

\[
\begin{align*}
  u(x,t) &= \pm ic_2 \sech \xi, \\
  v(x,t) &= \pm \frac{1}{2} ic_2 (c_1 - c_2) \tanh \xi \sech \xi, \\
  w(x,t) &= \frac{1}{4} c_2 (c_1 - c_2) \left( 1 - 2 \sech^2 \xi \right),
\end{align*}
\]

and also

\[
\begin{align*}
  u(x,t) &= \pm \frac{1}{2} ic_2 \left( \sech \xi + i \tanh \xi \right), \\
  v(x,t) &= \pm \frac{1}{4} c_2 (c_1 - c_2) \sech \xi \left( \sech \xi + i \tanh \xi \right), \\
  w(x,t) &= -\frac{1}{4} c_2 (c_1 - c_2) \sech \xi \left( \sech \xi + i \tanh \xi \right).
\end{align*}
\]

plus the c.c. solutions.

In all solutions $\xi = c_1 x + c_2 t + \Delta$. 
Algorithm for Jacobi Cn and Sn Solutions of PDEs

Given: System of nonlinear PDEs of order $m$

$$\Delta(u(x), u'(x), u''(x), \ldots, u^{(m)}(x)) = 0.$$  

Dependent variable $u$ has $M$ components $u_i$ (or $u, v, w, \ldots$).
Independent variable $x$ has $N$ components $x_j$ (or $x, y, z, \ldots, t$).

**Step CN1:**

- Seek solution $u(x) = U(CN)$, with

$$CN = \text{cn}(\xi; m) = \text{cn}\left(\sum_j c_j x_j + \Delta\right); m).$$  

with modulus $m$.

- Observe $\text{cn}'(\xi; m) = -\text{sn}(\xi; m) \text{dn}(\xi; m)$.

Using

$$\text{sn}^2(\xi; m) = 1 - \text{cn}^2(\xi; m), \quad \text{dn}^2(\xi; m) = 1 - m + m \text{cn}^2(\xi; m),$$  

one has

$$CN' = -\sqrt{(1 - CN^2)(1 - m + m CN^2)}.$$

- Repeatedly apply the operator rule

$$\frac{\partial \bullet}{\partial x_j} = \frac{d \bullet}{d CN} \frac{d CN}{d \xi} \frac{\partial \xi}{\partial x_j} = -c_j \sqrt{(1 - CN^2)(1 - m + m CN^2)} \frac{d \bullet}{d CN},$$

produces a nonlinear ODE:

$$\Delta(CN, U(CN), U'(CN), U''(CN), \ldots, U^{(m)}(CN)) = 0.$$
• Example: The KdV equation
\[ u_t + \alpha uu_x + u_{xxx} = 0, \]
transforms into
\[ \left( c_1^3(1 - 2m + 6m CN^2) - c_2 - \alpha c_1 U_1 \right) U_1' \\
+ 3c_1^3 CN(1 - 2m + 2m CN^2)U_1'' - c_1^3(1 - CN^2)(1 - m + m CN^2)U_1''' = 0. \]

**Step CN2:**

• Seek polynomial solutions
\[ U_i(CN) = \sum_{j=0}^{M_i} a_{ij} CN^j. \]

Determine the highest exponents \( M_i \geq 1. \)

• Example: For KdV case: \( M_1 = 2. \) Thus,
\[ U_1(CN) = a_{10} + a_{11} CN + a_{12} CN^2. \]

**Step CN3:**

• Derive the algebraic system for the coefficients \( a_{ij}. \)

• Example: Algebraic system for KdV case
\[-3 a_{11} c_1 (\alpha a_{12} - 2m c_1^2) = 0, \]
\[-2 a_{12} c_1 (\alpha a_{12} - 12m c_1^2) = 0, \]
\[-a_{11} (\alpha a_{10} c_1 - c_1^3 + 2m c_1^3 + c_2) = 0, \]
\[-\alpha a_{11}^2 c_1 - a_{12} (2 \alpha a_{10} c_1 - 16m c_1^3 - 8c_1^3 + 2c_2) = 0. \]

**Note:** modulus \( m \) is extra parameter.
Step CN4:

- Solve the nonlinear algebraic system with parameters.
- Example: Solution for KdV system

$$
\begin{align*}
  a_{10} &= \frac{4c_1^3 (1 - 2m) - c_2}{\alpha c_1}, \\
  a_{11} &= 0, \\
  a_{12} &= \frac{12m c_1^2}{\alpha}.
\end{align*}
$$

Step CN5:

- Return to the original variables. Test the final solution(s) of PDE. Reject trivial solutions.
- Example: Cnoidal solution for the KdV equation:

$$
u(x, t) = \frac{4c_1^3 (1 - 2m) - c_2}{\alpha c_1} + \frac{12m c_1^2}{\alpha} \text{cn}^2(c_1x + c_2t + \Delta; m).$$

**NOTE:** For Jacobi sn solutions, use

$$
\begin{align*}
  \text{cn}^2(\xi; m) &= 1 - \text{sn}^2(\xi; m), \\
  \text{dn}^2(\xi; m) &= 1 - m \text{sn}^2(\xi; m), \\
  \text{sn}'(\xi; m) &= \text{cn}(\xi; m) \text{dn}(\xi; m).
\end{align*}
$$

Hence,

$$
\text{SN}' = \sqrt{(1 - \text{SN}^2)(1 - m \text{SN}^2)},
$$

with $\text{SN} = \text{sn}(\xi; m)$.

Chain rule:

$$
\frac{\partial \bullet}{\partial x_j} = \frac{d \bullet}{d \text{SN}} \frac{d \text{SN}}{d \xi} \frac{\partial \xi}{\partial x_j} = c_j \sqrt{(1 - \text{SN}^2)(1 - m \text{SN}^2)} \frac{d \bullet}{d \text{SN}}.
$$
Algorithm for Tanh Solutions for system of DDEs

Given: System of nonlinear differential-difference equations (DDEs) of order \( m \)
\[
\Delta(..., u_{n-1}, u_n, u_{n+1}, ..., \dot{u}_n, ..., u_{n}^{(m)}) = 0.
\]

Dependent variable \( u_n \) has \( M \) components \( u_{i,n} \) (or \( u_n, v_n, w_n, ... \))
Independent variable \( x \) has 2 components \( x_i \) (or \( n, t \)).
No derivatives on shifted variables!

**Step D1:**

- Seek solution \( u_n(t) = U_n(T) \), with
  \[
  T = T_n(t) = \tanh [c_1 n + c_2 t + \Delta].
  \]

  - **Note:** The argument of \( T \) depends on \( n \).
- Repeatedly apply the operator rule
  \[
  \frac{d\bullet}{dt} = \frac{d\bullet}{dT} \frac{dT}{dt} = c_2 (1 - T^2) \frac{d\bullet}{dT}.
  \]
  Produces a nonlinear system of type
  \[
  \Delta(T, \cdots, U_{n-1}^{(i)}, U_n^{(i)}, U_{n+1}^{(i)}, \cdots, U_n^{(i)}, U_n^{''}, \cdots, U_{n}^{(m)}) = 0.
  \]
- **Example:** Toda lattice
  \[
  \ddot{u}_n = (1 + \dot{u}_n)(u_{n-1} - 2u_n + u_{n+1})
  \]
  transforms into
  \[
  c_2^2 (1 - T^2) [2TU_n' - (1 - T^2)U_n'' + [1 + c_2 (1 - T^2)U_n'] [U_n - 2U_n + U_{n+1}] = 0.
  \]
Step D2:

- Seek polynomial solutions
  \[ U_{i,n}(T_n) = \sum_{j=0}^{M_i} a_{ij} T_n^j. \]

Use
\[ \tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} \]
to deal with the shift:
\[ U_{i,n\pm p}(T(n \pm p)) = \sum_{j=0}^{M_i} a_{i,j} [T(n + p)]^j = \sum_{j=0}^{M_i} a_{i,j} \left[ \frac{T_n \pm \tanh(pc_1)}{1 \pm T_n \tanh(pc_1)} \right]^j. \]

Substitute \( U_{i,n} = T_n^{M_i} \), and
\[ U_{i,n\pm p}(T(n \pm p)) = [T(n + p)]^{M_i} = \left[ \frac{T_n \pm \tanh(pc_1)}{1 \pm T_n \tanh(pc_1)} \right]^{M_i}, \]
and balance the potential highest exponents in \( T_n \) to determine \( M_i \).

**Note:** \( U_{i,n\pm p}(T(n \pm p)) \) is homogeneous of degree zero in \( T \).

- Example: Balance of exponents for Toda lattice
  \[ 2M_1 - 1 = M_1 + 1. \]
  So, \( M_1 = 1 \).

Hence,
\[ U_n(T_n) = a_{10} + a_{11} T_n, \]
\[ U_{n\pm 1}(T(n \pm 1)) = a_{10} + a_{11} T(n \pm 1) = a_{10} + a_{11} \frac{T_n \pm \tanh(c_1)}{1 \pm T_n \tanh(c_1)}. \]
Step D3:

- Determine the algebraic system for the unknown coefficients $a_{ij}$ by setting to zero the coefficients of the powers in $T_n$.
- Example: Algebraic system for Toda lattice
  \[ c_2^2 - \tanh^2(c_1) - a_{11}c_2 \tanh^2(c_1) = 0, \]
  \[ c_2 - a_{11} = 0. \]

Step D4:

- Solve the nonlinear algebraic system with parameters.
- Example: Solution of algebraic system for Toda lattice
  \[ a_{10} = \text{free}, \]
  \[ a_{11} = \pm \sinh(c_1), \]
  \[ c_2 = \pm \sinh(c_1). \]

Step D5:

- Return to the original variables. Test solution(s) of DDE. Reject trivial ones.
- Example: Solitary wave solution for Toda lattice:
  \[ u_n(t) = a_0 \pm \sinh(c_1) \tanh [c_1n \pm \sinh(c_1) t + \Delta]. \]
Example: System of DDEs: Relativistic Toda lattice

\[ \dot{u}_n = (1 + \alpha u_n)(v_n - v_{n-1}), \]
\[ \dot{v}_n = v_n(u_{n+1} - u_n + \alpha v_{n+1} - \alpha v_{n-1}). \]

Change of variables

\[ u_n(t) = U_n(T_n), \quad v_n(t) = V_n(T_n), \]

with

\[ T_n(t) = \tanh [c_1 n + c_2 t + \Delta]. \]

gives

\[ c_2(1 - T^2)U'_n - (1 + \alpha U_n)(V_n - V_{n-1}) = 0, \]
\[ c_2(1 - T^2)V'_n - V_n(U_{n+1} - U_n + \alpha V_{n+1} - \alpha V_{n-1}) = 0. \]

Seek polynomial solutions

\[ U_n(T_n) = \sum_{j=0}^{M_1} a_{1j} T_n^j, \quad V_n(T_n) = \sum_{j=0}^{M_2} a_{2j} T_n^j. \]

Balance the highest exponents in \( T_n \) to determine \( M_1 \), and \( M_2 \) :

\[ M_1 + 1 = M_1 + M_2, \quad M_2 + 1 = M_1 + M_2. \]

So, \( M_1 = M_2 = 1. \) Hence,

\[ U_n = a_{10} + a_{11} T_n, \quad V_n = a_{20} + a_{21} T_n. \]

Algebraic system for \( a_{ij} \) :

\[ -a_{11} c_2 + a_{21} \tanh(c_1) + \alpha a_{10} a_{21} \tanh(c_1) = 0, \]
\[ a_{11} \tanh(c_1)(\alpha a_{21} + c_2) = 0, \]
\[ -a_{21} c_2 + a_{11} a_{20} \tanh(c_1) + 2\alpha a_{20} a_{21} \tanh(c_1) = 0, \]
\[ \tanh(c_1)(a_{11} a_{21} + 2\alpha a_{21}^2 - a_{11} a_{20} \tanh(c_1)) = 0, \]
\[ a_{21} \tanh^2(c_1)(c_2 - a_{11}) = 0. \]
Solution of the algebraic system

\[
\begin{align*}
a_{10} &= -c_2 \coth(c_1) - \frac{1}{\alpha}, \\
a_{11} &= c_2, \\
a_{20} &= \frac{c_2 \coth(c_1)}{\alpha}, \\
a_{21} &= -\frac{c_2}{\alpha}.
\end{align*}
\]

Solitary wave solution in original variables:

\[
\begin{align*}
u_n(t) &= -c_2 \coth(c_1) - \frac{1}{\alpha} + c_2 \tanh [c_1 n + c_2 t + \Delta], \\
v_n(t) &= \frac{c_2 \coth(c_1)}{\alpha} - \frac{c_2}{\alpha} \tanh [c_1 n + c_2 t + \Delta].
\end{align*}
\]
Analyzing and Solving Nonlinear Parameterized Systems

Assumptions:

- All \( c_i \neq 0 \) and modulus \( m \neq 0 \).
- Parameters \( (\alpha, \beta, \gamma, \ldots) \). Otherwise the maximal exponents \( M_i \) may change.
- All \( M_i \geq 1 \).
- All \( a_{iM_i} \neq 0 \). Highest power terms in \( U_i \) must be present, except in mixed sech-tanh-method.
- Solve for \( a_{ij} \), then \( c_i, m \) then find conditions on parameters.

Strategy followed by hand:

- Solve all linear equations in \( a_{ij} \) first (cost: branching). Start with the ones without parameters. Capture constraints in the process.
- Solve linear equations in \( c_i, m \) if they are free of \( a_{ij} \).
- Solve linear equations in parameters if they free of \( a_{ij}, c_i, m \).
- Solve quasi-linear equations for \( a_{ij}, c_i, m \) parameters.
- Solve quadratic equations for \( a_{ij}, c_i, m \) parameters.
- Eliminate cubic terms for \( a_{ij}, c_i, m \) parameters, without solving.
- Show remaining equations, if any.

Alternatives:

- Use (adapted) Gröbner bases techniques.
- Use Ritt-Wu characteristic sets method.
- Use combinatorics on coefficients \( a_{ij} = 0 \) or \( a_{ij} \neq 0 \).
Implementation Issues – Software Demo – Future Work

• Demonstration of Mathematica package for hyperbolic and elliptic function methods for PDEs and DDEs.

• Long term goal: Develop PDESolve and DDESolve for analytical solutions of nonlinear PDEs and DDEs.

• Implement various methods: Lie symmetry methods, etc.

• Look at other types of explicit solutions involving
  – other hyperbolic and elliptic functions sinh, cosh, dn, ....
  – complex exponentials combined with sech or tanh.

• Seek solutions $u(x, t) = U(F(\xi))$, for special functions $F$, where $F'(\xi)$ is polynomial or irrational expression in $F$.

Examples:
  – If $F = \tanh \xi$
    
    $F'(\xi) = 1 - F^2(\xi)$.

    Chain rule:
    
    $\frac{\partial \bullet}{\partial x_j} = c_j(1 - F^2) \frac{d\bullet}{dF}$.

  – If $F = \sech \xi$
    
    $F'(\xi) = -F(\xi) \sqrt{1 - F^2(\xi)}$.

    Chain rule:
    
    $\frac{\partial \bullet}{\partial x_j} = -c_j F \sqrt{1 - F^2} \frac{d\bullet}{dF}$.

  – If $F = \cn \xi$
    
    $cn' \xi = -sn \xi \ dn \xi$
    
    $F'(\xi) = -\sqrt{1 - F^2} \sqrt{1 - m + mF^2}$. 
Chain rule:
\[
\frac{\partial \bullet}{\partial x_j} = -c_j \sqrt{1 - F^2} \sqrt{1 - m + mF^2} \frac{d\bullet}{dF}.
\]

• Add the constraining differential equations to the system of PDEs directly.

• Why are tanh and sech solutions so prevalent?

• Other applications (of the nonlinear algebraic solver):
  Computation of conservation laws, symmetries, first integrals, etc. leading to linear parameterized systems for unknowns coefficients (see InvariantsSymmetries by Göktaş and Hereman).
• Preprint:
  Available from http://www.mines.edu/fs_home/whereman/

• Software:
  Available via anonymous FTP from mines.edu in directory pub/papers/math_cs_dept/software/pde-sols;
or via Internet URL: http://www.mines.edu/fs_home/whereman/

  Available via anonymous FTP from mines.edu in directory pub/papers/math_cs_dept/software/dde-sols;
or via Internet URL: http://www.mines.edu/fs_home/whereman/
Appendix: A Complicated Case

Class of fifth-order evolution equations with parameters:

\[ u_t + \alpha\gamma^2 u^2 u_x + \beta\gamma u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0. \]

Well-Known Special cases

Lax case: \( \alpha = \frac{3}{10}, \beta = 2, \gamma = 10 \). Two solutions:

\[ u(x, t) = 4c_1^2 - 6c_1^2 \tanh^2 \left[ c_1 x - 56c_1^5 t + \Delta \right], \]

and

\[ u(x, t) = a_0 - 2c_1^2 \tanh^2 \left[ c_1 x - 2(15a_0^2 c_1 - 40a_0 c_1^3 + 28c_1^5) t + \Delta \right], \]

where \( a_0 \) is arbitrary.

Sawada-Kotera case: \( \alpha = \frac{1}{5}, \beta = 1, \gamma = 5 \). Two solutions:

\[ u(x, t) = 8c_1^2 - 12c_1^2 \tanh^2 \left[ c_1 x - 16c_1^5 t + \Delta \right], \]

and

\[ u(x, t) = a_0 - 6c_1^2 \tanh^2 \left[ c_1 x - (5a_0^2 c_1 - 40a_0 c_1^3 + 76c_1^5) t + \Delta \right], \]

where \( a_0 \) is arbitrary.

Kaup-Kupershmidt case: \( \alpha = \frac{1}{5}, \beta = \frac{5}{2}, \gamma = 10 \). Two solutions:

\[ u(x, t) = c_1^2 - \frac{3}{2} c_1^2 \tanh^2 \left[ c_1 x - c_1^5 t + \Delta \right], \]

and

\[ u(x, t) = 8c_1^2 - 12c_1^2 \tanh^2 \left[ c_1 x - 176c_1^5 t + \Delta \right]. \]

No free constants!

Ito case: \( \alpha = \frac{2}{5}, \beta = 2, \gamma = 3 \). One solution:

\[ u(x, t) = 20c_1^2 - 30c_1^2 \tanh^2 \left[ c_1 x - 96c_1^5 t + \Delta \right]. \]
What about the General case?

Q1: Can we retrieve the special solutions?
Q2: What are the condition(s) on the parameters $\alpha, \beta, \gamma$ for solutions of tanh-type to exist?

Tanh solutions:

$$u(x, t) = a_0 + a_1 \tanh [c_1x + c_2t + \Delta] + a_2 \tanh^2 [c_1x + c_2t + \Delta].$$

Nonlinear algebraic system must be analyzed, solved (or reduced!):

$$a_1(\alpha \gamma^2 a_2^2 + 6\gamma a_2 c_1^2 + 2\beta \gamma a_2 c_1^2 + 24c_1^4) = 0,$$

$$a_1(\alpha \gamma^2 a_1^2 + 6\alpha \gamma^2 a_0 a_2 + 6\gamma a_0 c_1^2 - 18\gamma a_2 c_1^2 - 12\beta \gamma a_2 c_1^2 - 120c_1^4) = 0,$$

$$\alpha \gamma^2 a_2^2 + 12\gamma a_2 c_1^2 + 6\beta \gamma a_2 c_1^2 + 360c_1^4 = 0,$$

$$2\alpha \gamma^2 a_1^2 a_2 + 2\alpha \gamma^2 a_0 a_2^2 + 3\gamma a_1^2 c_1^2 + \beta \gamma a_1^2 c_1^2 + 12\gamma a_0 a_2 c_1^2 - 8\gamma a_2 c_1^2 - 8\beta \gamma a_2^2 c_1^2 - 480a_2 c_1^4 = 0,$$

$$a_1(\alpha \gamma^2 a_0^2 c_1 - 2\gamma a_0 c_1^3 + 2\beta \gamma a_2 c_1^3 + 16c_1^5 + c_2) = 0,$$

$$\alpha \gamma^2 a_0 a_2^2 c_1 - \alpha \gamma^2 a_0^2 c_1 + \gamma a_1^2 c_1^3 - \beta \gamma a_1^2 c_1^3 - 8\gamma a_0 a_2 c_1^3 + 2\beta \gamma a_2 c_1^3 + 136a_2 c_1^5 + a_2 c_2 = 0.$$

Unknowns: $a_0, a_1, a_2$.

Parameters: $c_1, c_2, \alpha, \beta, \gamma$.

**Solve** and **Reduce** cannot be used on the whole system!
**Actual Solution:** Two major cases:

CASE 1: $a_1 = 0$, two subcases

**Subcase 1-a:**

\[
a_2 = -\frac{3}{2}a_0,
\]

\[
c_2 = c_1^3(24c_1^2 - \beta \gamma a_0),
\]

where $a_0$ is one of the two roots of the quadratic equation:

\[
\alpha \gamma^2 a_0^2 - 8\gamma a_0 c_1^2 - 4\beta \gamma a_0 c_1^2 + 160c_1^4 = 0.
\]

**Subcase 1-b:** If $\beta = 10\alpha - 1$, then

\[
a_2 = -\frac{6}{\alpha \gamma} c_1^2,
\]

\[
c_2 = -\frac{1}{\alpha}(\alpha^2 \gamma^2 a_0^2 c_1 - 8\alpha \gamma a_0 c_1^3 + 12c_1^5 + 16\alpha c_1^5),
\]

where $a_0$ is arbitrary.

CASE 2: $a_1 \neq 0$, then

\[
\alpha = \frac{1}{392}(39 + 38\beta + 8\beta^2)
\]

and

\[
a_2 = -\frac{168}{\gamma(3 + 2\beta)} c_1^2,
\]

provided $\beta$ is root of

\[
(104\beta^2 + 886\beta + 1487)(520\beta^3 + 2158\beta^2 - 1103\beta - 8871) = 0.
\]
Subcase 2-a: If $\beta^2 = -\frac{1}{104}(886\beta + 1487)$, then

\[
\alpha = -\frac{2\beta + 5}{26},
\]

\[
a_0 = -\frac{49c_1^2(9983 + 4378\beta)}{26\gamma(8 + 3\beta)(3 + 2\beta)^2},
\]

\[
a_1 = \pm \frac{336c_1^2}{\gamma(3 + 2\beta)},
\]

\[
a_2 = -\frac{168c_1^2}{\gamma(3 + 2\beta)},
\]

\[
c_2 = -\frac{364c_1^5(3851 + 1634\beta)}{6715 + 2946\beta}.
\]

Subcase 2-b: If $\beta^3 = \frac{1}{520}(8871 + 1103\beta - 2158\beta^2)$, then

\[
\alpha = \frac{39 + 38\beta + 8\beta^2}{392},
\]

\[
a_0 = \frac{28c_1^2 (6483 + 5529\beta + 1066\beta^2)}{(3 + 2\beta)(23 + 6\beta)(81 + 26\beta)\gamma},
\]

\[
a_1^2 = \frac{28224c_1^4 (4\beta - 1)(26\beta - 17)}{(3 + 2\beta)^2(23 + 6\beta)(81 + 26\beta)\gamma^2},
\]

\[
a_2 = -\frac{168c_1^2}{\gamma(3 + 2\beta)},
\]

\[
c_2 = -\frac{8c_1^5 (1792261977 + 1161063881\beta + 188900114\beta^2)}{959833473 + 632954969\beta + 105176786\beta^2}.
\]