Symbolic Computation of
Conserved Densities and Generalized Symmetries for
Nonlinear Evolution and Lattice Equations

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• **Purpose**

Design and implement algorithms to compute polynomial conservation laws and generalized symmetries (later recursion operators) for nonlinear systems of evolution and lattice equations.

• **Motivation**

  – Conservation laws describe the conservation of fundamental physical quantities (linear momentum, energy, etc.). Compare with constants of motion (linear momentum, energy) in mechanics.

  – Conservation laws provide a method to study quantitative and qualitative properties of equations and their solutions, e.g. Hamiltonian structures.

  – Conservation laws can be used to test numerical integrators.

  – For PDEs and DDEs, the existence of a sufficiently large (in principal infinite) number of conservation laws or symmetries assures complete **integrability**.

  – Conserved densities and symmetries aid in finding the recursion operator (which guarantees the existence of infinitely many symmetries).
PART I: Evolution Equations (PDEs)

- **System of evolution equations**

  \[ u_t = F(u, u_x, u_{2x}, \ldots, u_{mx}) \]

  in a (single) space variable \( x \) and time \( t \), and with

  \[ u = (u_1, u_2, \ldots, u_n), \quad F = (F_1, F_2, \ldots, F_n). \]

  Notation:

  \[ u_{mx} = u^{(m)} = \frac{\partial u}{\partial x^m}. \]

  \( F \) is polynomial in \( u, u_x, \ldots, u_{mx} \).

  PDEs of higher order in \( t \) should be recast as a first-order system.

- **Examples:**

  The Korteweg-de Vries (KdV) equation:

  \[ u_t + uu_x + u_{3x} = 0. \]

  Fifth-order evolution equations with constant parameters \((\alpha, \beta, \gamma)\):

  \[ u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma uu_{3x} + u_{5x} = 0. \]

  Special case. The fifth-order Sawada-Kotera (SK) equation:

  \[ u_t + 5u^2 u_x + 5u_x u_{2x} + 5uu_{3x} + u_{5x} = 0. \]

  The Boussinesq (wave) equation:

  \[ u_{tt} - u_{2x} + 3uu_{2x} + 3u_x^2 + \alpha u_{4x} = 0, \]

  written as a first-order system (\( v \) auxiliary variable):

  \[ u_t + v_x = 0, \]

  \[ v_t + u_x - 3uu_x - \alpha u_{3x} = 0. \]
A vector nonlinear Schrödinger equation:
\[
B_t + (|B|^2 B)_x + (B_0 \cdot B_x)B_0 + e \times B_{xx} = 0,
\]
written in component form, \(B_0 = (a, b)\) and \(B = (u, v)\):
\[
\begin{align*}
  u_t + [u(u^2 + v^2) + \beta u + \gamma v - v_x]_x &= 0, \\
v_t + [v(u^2 + v^2) + \theta u + \delta v + u_x]_x &= 0,
\end{align*}
\]
\(\beta = a^2, \gamma = \theta = ab,\) and \(\delta = b^2.\)

- **Key concept: Dilation invariance.**

Conservation laws, symmetries and recursion operators are invariant under the dilation (scaling) symmetry of the given PDE.

The KdV equation, \(u_t + uu_x + u_{3x} = 0,\) has scaling symmetry
\[
(t, x, u) \rightarrow (\lambda^{-3} t, \lambda^{-1} x, \lambda^2 u).
\]

\(u\) corresponds to two \(x\)-derivatives, \(u \sim D_x^2.\) Similarly, \(D_t \sim D_x^3.\)

The **weight**, \(w,\) of a variable equals the number of \(x\)-derivatives the variable carries.

Weights are rational. Weights of dependent variables are nonnegative.

Set \(w(D_x) = 1.\)

Due to dilation invariance: \(w(u) = 2\) and \(w(D_t) = 3.\)

Consequently, \(w(x) = -1\) and \(w(t) = -3.\)

The **rank** of a monomial is its total weight in terms of \(x\)-derivatives.
Every monomial in the KdV equation has rank 5.
The KdV equation is \textit{uniform in rank}.

What do we do if equations are not uniform in rank?
Extend the space of dependent variables with parameters carrying weight.
Example: the Boussinesq system
\begin{align*}
u_t + v_x &= 0, \\
v_t + u_x - 3uu_x - \alpha u_{3x} &= 0,
\end{align*}
is not scaling invariant ($u_x$ and $u_{3x}$ are conflict terms).
Introduce an auxiliary parameter $\beta$
\begin{align*}
u_t + v_x &= 0, \\
v_t + \beta u_x - 3uu_x - \alpha u_{3x} &= 0,
\end{align*}
which has scaling symmetry:
\[(x, t, u, v, \beta) \rightarrow (\lambda x, \lambda^2 t, \lambda^{-2} u, \lambda^{-3} v, \lambda^{-2} \beta).\]

\textbf{CONSERVATION LAWS.}
\[D_t \rho + D_x J = 0,\]
with conserved density $\rho$ and flux $J$.
Both are polynomial in $u, u_x, u_{2x}, u_{3x}, \ldots$
\[P = \int_{-\infty}^{+\infty} \rho \, dx = \text{constant}\]
if $J$ vanishes at infinity.
Conserved densities are equivalent if they differ by a $D_x$ term.
**Example:** The Korteweg-de Vries (KdV) equation

\[ u_t + uu_x + u_{3x} = 0. \]

Conserved densities:

\[ \rho_1 = u, \quad D_t(u) + D_x\left(\frac{u^2}{2} + u_{2x}\right) = 0. \]

\[ \rho_2 = u^2, \quad D_t(u^2) + D_x\left(\frac{2u^3}{3} + 2uu_{2x} - u_x^2\right) = 0. \]

\[ \rho_3 = u^3 - 3u_x^2, \quad D_t\left(u^3 - 3u_x^2\right) + D_x\left(\frac{3}{4}u^4 - 6uu_x^2 + 3u^2u_{2x} + 3u_{2x}^2 - 6u_xu_{3x}\right) = 0. \]

\[ \vdots \]

\[ \rho_6 = u^6 - 60u^3u_x^2 - 30u_x^4 + 108u^2u_{2x}^2 \]

\[ + \frac{720}{7}u_{2x}^3 - \frac{648}{7}uu_{3x}^2 + \frac{216}{7}u_{4x}^2. \]

Time and space dependent conservation law:

\[ D_t\left(tu^2 - 2xu\right) + D_x\left(\frac{2}{3}tu^3 - xu^2 + 2tuu_{2x} - tu_x^2 - 2xu_{2x} + 2u_x\right) = 0. \]

**Algorithm for Conservation Laws of PDEs.**

1. Determine weights (scaling properties) of variables and auxiliary parameters.

2. Construct the form of the density (find monomial building blocks).

3. Determine the constant coefficients.
• **Example:** Density of rank 6 for the KdV equation.

**Step 1: Compute the weights.**

Require uniformity in rank. With \( w(D_x) = 1 \):

\[
w(u) + w(D_t) = 2w(u) + 1 = w(u) + 3.
\]

Solve the linear system: \( w(u) = 2 \), \( w(D_t) = 3 \).

**Step 2: Determine the form of the density.**

List all possible powers of \( u \), up to rank 6 : \([u, u^2, u^3]\).

Introduce \( x \) derivatives to ‘complete’ the rank.

- \( u \) has weight 2, introduce \( D_x^4 \).
- \( u^2 \) has weight 4, introduce \( D_x^2 \).
- \( u^3 \) has weight 6, no derivative needed.

Apply the \( D_x \) derivatives.

Remove terms of the form \( D_x u_{px} \), or \( D_x \) up to terms kept prior in the list.

\[
[u_{4x}] \rightarrow [\ ] \quad \text{empty list.}
\]

\[
[u_x^2, uu^2] \rightarrow [u_x^2] \quad \text{since} \quad uu^2 = (uu_x)_x - u_x^2.
\]

\[
[u^3] \rightarrow [u^3].
\]

Linearly combine the ‘building blocks’:

\[
\rho = c_1 u^3 + c_2 u_x^2.
\]
Step 3: Determine the coefficients $c_i$.

Compute $D_t \rho = 3c_1 u^2 u_t + 2c_2 u_x u_{xt}$.

Replace $u_t$ by $-(uu_x + u_3 x)$ and $u_{xt}$ by $-(uu_x + u_3 x)_x$.

Integrate the result, $E$, with respect to $x$. To avoid integration by parts, apply the Euler operator (variational derivative)

$$L_u = \sum_{k=0}^{m} (-D_x)^k \frac{\partial}{\partial u_{kx}}$$

$$= \frac{\partial}{\partial u} - D_x \left( \frac{\partial}{\partial u_x} \right) + D_x^2 \left( \frac{\partial}{\partial u_{2x}} \right) + \cdots + (-1)^m D_x^m \left( \frac{\partial}{\partial u_{mx}} \right).$$

to $E$ of order $m$.

If $L_u(E) = 0$ immediately, then $E$ is a total $x$-derivative.

If $L_u(E) \neq 0$, the remaining expression must vanish identically.

$$D_t \rho = -D_x \left[ \frac{3}{4} c_1 u^4 - (3c_1 - c_2) uu_x^2 + 3c_1 u^2 u_{2x} - c_2 u_{2x}^2 + 2c_2 u_x u_{3x} \right] - (3c_1 + c_2) u_x^3.$$

The non-integrable term must vanish.

So, $c_1 = -\frac{1}{3} c_2$. Set $c_2 = -3$, hence, $c_1 = 1$.

Result:

$$\rho = u^3 - 3u_x^2.$$

Expression $[...]$ yields

$$J = \frac{3}{4} u^4 - 6uu_x^2 + 3u^2 u_{2x} + 3u_{2x}^2 - 6u_x u_{3x}.$$

Example: First few densities for the Boussinesq system:

$$\rho_1 = u, \quad \rho_2 = v, \quad \rho_3 = uv, \quad \rho_4 = \beta u^2 - u^3 + v^2 + \alpha u_x^2.$$

(then substitute $\beta = 1$).
• Application.

A Class of Fifth-Order Evolution Equations

\[ u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma uu_{3x} + u_{5x} = 0 \]

where \( \alpha, \beta, \gamma \) are nonzero parameters.

\[ u \sim D_x^2. \]

Special cases:

\[ \begin{align*}
\alpha &= 30 & \beta &= 20 & \gamma &= 10 & \text{Lax.} \\
\alpha &= 5 & \beta &= 5 & \gamma &= 5 & \text{Sawada – Kotera.} \\
\alpha &= 20 & \beta &= 25 & \gamma &= 10 & \text{Kaup–Kupershmidt.} \\
\alpha &= 2 & \beta &= 6 & \gamma &= 3 & \text{Ito.}
\end{align*} \]

What are the conditions for the parameters \( \alpha, \beta \) and \( \gamma \) so that the equation admits a density of fixed rank?

- **Rank 2:**
  No condition
  \[ \rho = u. \]

- **Rank 4:**
  Condition: \( \beta = 2\gamma \) (Lax and Ito cases)
  \[ \rho = u^2. \]
– Rank 6:
Condition:
\[10\alpha = -2\beta^2 + 7\beta\gamma - 3\gamma^2\]
(Lax, SK, and KK cases)
\[\rho = u^3 + \frac{15}{(-2\beta + \gamma)}u_x^2.\]

– Rank 8:

1. \(\beta = 2\gamma\) (Lax and Ito cases)
\[\rho = u^4 - \frac{6\gamma}{\alpha}uu_x^2 + \frac{6}{\alpha}u_{2x}^2.\]

2. \(\alpha = -\frac{2\beta^2 - 7\beta\gamma - 4\gamma^2}{45}\) (SK, KK and Ito cases)
\[\rho = u^4 - \frac{135}{2\beta + \gamma}uu_x^2 + \frac{675}{(2\beta + \gamma)^2}u_{2x}^2.\]

– Rank 10:

Condition:
\[\beta = 2\gamma\]
and
\[10\alpha = 3\gamma^2\]
(Lax case)
\[\rho = u^5 - \frac{50}{\gamma}u^2u_x^2 + \frac{100}{\gamma^2}u_{uu}^2 - \frac{500}{7\gamma^3}u_{3x}^2.\]
What are the necessary conditions for the parameters $\alpha$, $\beta$ and $\gamma$ so that the equation admits $\infty$ many polynomial conservation laws?

- If $\alpha = \frac{3}{10} \gamma^2$ and $\beta = 2\gamma$ then there is a sequence (without gaps!) of conserved densities (Lax case).

- If $\alpha = \frac{1}{5} \gamma^2$ and $\beta = \gamma$ then there is a sequence (with gaps!) of conserved densities (SK case).

- If $\alpha = \frac{1}{5} \gamma^2$ and $\beta = \frac{5}{2} \gamma$ then there is a sequence (with gaps!) of conserved densities (KK case).

- If

  $$\alpha = -\frac{2\beta^2 - 7\beta \gamma + 4\gamma^2}{45}$$

  or

  $$\beta = 2\gamma$$

  then there is a conserved density of rank 8.

Combine both conditions: $\alpha = \frac{2\gamma^2}{9}$ and $\beta = 2\gamma$ (Ito case).

SUMMARY: see tables (notice the gaps!)
• **GENERALIZED SYMMETRY.**

\[ G(x, t, u, u_x, u_{2x}, ...) \]

with \( G = (G_1, G_2, ..., G_n) \) is a symmetry iff it leaves the PDE invariant for the replacement \( u \rightarrow u + \epsilon G \) within order \( \epsilon \). i.e.

\[ D_t(u + \epsilon G) = F(u + \epsilon G) \]

must hold up to order \( \epsilon \) on the solutions of PDE.

Consequently, \( G \) must satisfy the linearized equation

\[ D_t G = F'(u)[G], \]

where \( F' \) is the Fréchet derivative of \( F \), i.e.,

\[ F'(u)[G] = \frac{\partial}{\partial \epsilon} F(u + \epsilon G) |_{\epsilon=0}. \]

Here \( u \) is replaced by \( u + \epsilon G \), and \( u_{nx} \) by \( u_{nx} + \epsilon D_x^n G \).

• **Example.**

Consider the KdV equation

\[ u_t = 6uu_x + u_{3x}. \]

Generalized symmetries:

\[ G^{(1)} = u_x, \quad G^{(2)} = 6uu_x + u_{3x}, \]
\[ G^{(3)} = 30u^2u_x + 20u_xu_{2x} + 10uu_3x + u_{5x}, \]
\[ G^{(4)} = 140u^3u_x + 70u^3_x + 280uu_xu_{2x} + 70u^2u_{3x} + 70u_{2x}u_{3x} + 42u_xu_{4x} + 14uu_5x + u_7x. \]
Algorithm for Generalized Symmetries of PDEs.

Consider the KdV equation, \( u_t = 6uu_x + u_{3x} \), with \( w(u) = 2 \).

**Step 1: Construct the form of the symmetry.**

Compute the form of the symmetry with rank 7.

List all powers in \( u \) with rank 7 or less:

\[ \mathcal{L} = \{1, u, u^2, u^3\}. \]

For each monomial in \( \mathcal{L} \), introduce the needed \( x \)-derivatives, so that each term exactly has rank 7. Thus,

\[
\begin{align*}
D_x(u^3) &= 3u^2u_x, \\
D_x^3(u^2) &= 6u_xu_{2x} + 2uu_{3x}, \\
D_x^5(u) &= u_{5x}, \\
D_x^7(1) &= 0.
\end{align*}
\]

Gather the resulting (non-zero) terms

\[ \mathcal{R} = \{u^2u_x, u_xu_{2x}, uu_{3x}, u_{5x}\}. \]

The symmetry is a linear combination of these monomials:

\[ G = c_1 u^2u_x + c_2 u_xu_{2x} + c_3 uu_{3x} + c_4 u_{5x}. \]

**Step 2: Determine the unknown coefficients \( c_i \).**

Compute \( D_tG \) and use KdV to remove \( u_t, u_{tx}, u_{txx} \), etc.

Compute the Fréchet derivative.

Equate the resulting expressions.

Group the terms:

\[
(12c_1 - 18c_2)u_x^2u_{2x} + (6c_1 - 18c_3)uu_{2x}^2 + (6c_1 - 18c_3)uu_xu_{3x} + (3c_2 - 60c_4)u_{3x}^2 + (3c_2 + 3c_3 - 90c_4)u_{2x}u_{4x} + (3c_3 - 30c_4)u_xu_{5x} \equiv 0.
\]
Solve the linear system:

\[
S = \{ 12c_1 - 18c_2 = 0, 6c_1 - 18c_3 = 0, 3c_2 - 60c_4 = 0, \\
3c_2 + 3c_3 - 90c_4 = 0, 3c_3 - 30c_4 = 0 \}.
\]

Solution: \( \frac{c_1}{30} = \frac{c_2}{20} = \frac{c_3}{10} = c_4 \).

Setting \( c_4 = 1 \) one gets: \( c_1 = 30, c_2 = 20, c_3 = 10 \).

Substitute the result into the symmetry:

\[
G = 30u^2 u_x + 20uu_x u_{2x} + 10uu_{3x} + uu_{5x}.
\]

Note that \( u_t = G \) is known as the Lax equation.

- **x-t Dependent symmetries.**

The KdV equation has also symmetries which explicitly depend on \( x \) and \( t \).

The same algorithm can be used provided the highest degree of \( x \) or \( t \) is specified.

Compute the symmetry of rank 2, that is linear in \( x \) or \( t \).

List all monomials in \( u, x \) and \( t \) of rank 2 or less:

\[
\mathcal{L} = \{ 1, u, x, xu, t, tu, tu^2 \}.
\]

For each monomial in \( \mathcal{L} \), introduce enough \( x \)-derivatives, so that each term exactly has rank 2. Thus,

\[
D_x(xu) = u + xu_x, \quad D_x(tu^2) = 2tuu_x, \quad D_x^3(tu) = tu_{3x}, \\
D_x^2(1) = D_x^3(x) = D_x^5(t) = 0.
\]

Gather the non-zero resulting terms:

\[
\mathcal{R} = \{ u, xu_x, tuu_x, tu_{3x} \},
\]

Build the linear combination

\[
G = c_1u + c_2xu_x + c_3tuu_x + c_4tu_{3x}.
\]
Determine the coefficients $c_1$ through $c_4$:

$$G = \frac{2}{3}u + \frac{1}{3}xu_x + 6tu_{ux} + tu_{3x}.$$ 

Two symmetries of KdV that explicitly depend on $x$ and $t$:

$$G = 1 + 6tu_x, \text{ and } G = 2u + xu_x + 3t(6uu_x + u_{3x}),$$

of rank 0 and 2, respectively.
PART II: Differential-difference (lattice) Equations

• Systems of lattices equations

Consider the system of lattice equations, continuous in time, discretized in (one dimensional) space

\[ \dot{u}_n = F(..., u_{n-1}, u_n, u_{n+1}, ...) \]

where \( u_n \) and \( F \) are vector dynamical variables.
\( F \) is polynomial with constant coefficients.
No restrictions on the level of the shifts or the degree of nonlinearity.

• CONSERVATION LAW:

\[ \dot{\rho}_n = J_n - J_{n+1} \]

with density \( \rho_n \) and flux \( J_n \).
Both are polynomials in \( u_n \) and its shifts.

\[ \frac{d}{dt} (\sum_n \rho_n) = \sum_n \dot{\rho}_n = \sum_n (J_n - J_{n+1}) \]

if \( J_n \) is bounded for all \( n \).
Subject to suitable boundary or periodicity conditions

\[ \sum_n \rho_n = \text{constant}. \]
**Example.**

Consider the one-dimensional Toda lattice

\[ \ddot{y}_n = \exp (y_{n-1} - y_n) - \exp (y_n - y_{n+1}) \]

\( y_n \) is the displacement from equilibrium of the \( n \)th particle with unit mass under an exponential decaying interaction force between nearest neighbors.

Change of variables:

\[ u_n = \dot{y}_n, \quad v_n = \exp (y_n - y_{n+1}) \]

yields

\[ \dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n (u_n - u_{n+1}). \]

Toda system is completely integrable.

The first two density-flux pairs (computed by hand):

\[ \rho_n^{(1)} = u_n, \quad J_n^{(1)} = v_{n-1}, \quad \text{and} \quad \rho_n^{(2)} = \frac{1}{2} u_n^2 + v_n, \quad J_n^{(2)} = u_n v_{n-1}. \]

**Key concept: Dilation invariance.**

The Toda system as well as the conservation laws and symmetries are invariant under the dilation symmetry

\[ (t, u_n, v_n) \to (\lambda^{-1} t, \lambda u_n, \lambda^2 v_n). \]

Thus, \( u_n \) corresponds to one \( t \)-derivative: \( u_n \sim \frac{d}{dt} \).

Similarly, \( v_n \sim \frac{d^2}{dt^2} \).

**Weight, w, of variables are defined in terms of t-derivatives.**

Set \( w(\frac{d}{dt}) = 1. \)
Weights of dependent variables are nonnegative, rational, and independent of $n$.

Due to dilation invariance: $w(u_n) = 1$ and $w(v_n) = 2$.

The rank of a monomial is its total weight in terms of $t$-derivatives.

Require uniformity in rank for each equation to compute the weights:

(solve the linear system):

$$w(u_n) + 1 = w(v_n), \quad w(v_n) + 1 = w(u_n) + w(v_n),$$

Solving the linear system yields $w(u_n) = 1, w(v_n) = 2$.

• Equivalence Criterion.

Define $D$ shift-down operator, and $U$ shift-up operator, on the set of all monomials in $u_n$ and their shifts.

For a monomial $m$:

$$Dm = m|_{n \rightarrow n-1}, \quad \text{and} \quad Um = m|_{n \rightarrow n+1}.$$ 

For example

$$Du_{n+2}v_n = u_{n+1}v_{n-1}, \quad Uu_{n-2}v_{n-1} = u_{n-1}v_n.$$ 

Compositions of $D$ and $U$ define an equivalence relation. All shifted monomials are equivalent.

For example

$$u_{n-1}v_{n+1} \equiv u_{n+2}v_{n+4} \equiv u_{n-3}v_{n-1}.$$
Equivalence criterion:

Two monomials $m_1$ and $m_2$ are equivalent, $m_1 \equiv m_2$, if

$$m_1 = m_2 + [M_n - M_{n+1}]$$

for some polynomial $M_n$.

For example, $u_{n-2}u_n \equiv u_{n-1}u_{n+1}$ since

$$u_{n-2}u_n = u_{n-1}u_{n+1} + [u_{n-2}u_n - u_{n-1}u_{n+1}] = u_{n-1}u_{n+1} + [M_n - M_{n+1}].$$

Main representative of an equivalence class is the monomial with label $n$ on $u$ (or $v$).

For example, $u_n u_{n+2}$ is the main representative of the class with elements $u_{n-1}u_{n+1}, u_{n+1}u_{n+3},$ etc.

Use lexicographical ordering to resolve conflicts.

For example, $u_n v_{n+2}$ (not $u_{n-2}v_n$) is the main representative of the class with elements $u_{n-3}v_{n-1}, u_{n+2}v_{n+4},$ etc.

• Steps of the Algorithm for Lattices.

Three-step algorithm to find conserved densities:

1. Determine the weights.
2. Construct the form of density.
3. Determine the coefficients.
Example: Density of rank 3 or the Toda lattice,

\[ \dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}) . \]

**Step 1: Compute the weights.**

Here \( w(u_n) = 1 \) and \( w(v_n) = 2 \).

**Step 2: Construct the form of the density.**

List all monomials in \( u_n \) and \( v_n \) of rank 3 or less:

\[ \mathcal{G} = \{ u_n^3, u_n^2, u_n v_n, u_n, v_n \} . \]

For each monomial in \( \mathcal{G} \), introduce enough \( t \)-derivatives to obtain weight 3. Use the lattice to remove \( \dot{u}_n \) and \( \dot{v}_n \):

\[
\begin{align*}
\frac{d^0}{dt^0}(u_n^3) &= u_n^3, \\
\frac{d^0}{dt^0}(u_n v_n) &= u_n v_n, \\
\frac{d}{dt}(u_n^2) &= 2u_n v_{n-1} - 2u_n v_n, \\
\frac{d}{dt}(v_n) &= u_n v_n - u_{n+1} v_n, \\
\frac{d^2}{dt^2}(u_n) &= u_{n-1} v_{n-1} - u_n v_{n-1} - u_n v_n + u_{n+1} v_n .
\end{align*}
\]

Gather the resulting terms in a set

\[ \mathcal{H} = \{ u_n^3, u_n v_{n-1}, u_n v_n, u_{n-1} v_{n-1}, u_{n+1} v_n \} . \]

Replace members in the same equivalence class by their main representatives.

For example, \( u_n v_{n-1} \equiv u_{n+1} v_n \) are replaced by \( u_n v_{n-1} \).

Linearly combine the monomials in

\[ \mathcal{I} = \{ u_n^3, u_n v_{n-1}, u_n v_n \} \]

to obtain

\[ \rho_n = c_1 u_n^3 + c_2 u_n v_{n-1} + c_3 u_n v_n . \]
Step 3: Determine the coefficients.

Require that $\dot{\rho}_n = J_n - J_{n+1}$, holds.

Compute $\dot{\rho}_n$ and use the lattice to remove $\dot{u}_n$ and $\dot{v}_n$.

Group the terms

$$\dot{\rho}_n = (3c_1 - c_2)u_n^2v_{n-1} + (c_3 - 3c_1)u_n^2v_n + (c_3 - c_2)v_{n-1}v_n + c_2u_{n-1}u_nv_{n-1} + c_2v_{n-1}^2 - c_3u_nu_{n+1}v_n - c_3v_n^2.$$

Use the equivalence criterion to modify $\dot{\rho}_n$.

Replace $u_{n-1}u_nv_{n-1}$ by $u_nu_{n+1}v_n + [u_{n-1}u_nv_{n-1} - u_nu_{n+1}v_n]$.

Introduce the main representatives. Thus

$$\dot{\rho}_n = (3c_1 - c_2)u_n^2v_{n-1} + (c_3 - 3c_1)u_n^2v_n + (c_3 - c_2)v_{n-1}v_n + c_2u_{n-1}u_nv_{n-1} - c_3u_nu_{n+1}v_n - c_3v_n^2.$$

Group the terms outside of the square brackets and move the pairs inside the square brackets to the bottom.

Rearrange the terms to match the pattern $[J_n - J_{n+1}]$.

Hence

$$\dot{\rho}_n = (3c_1 - c_2)u_n^2v_{n-1} + (c_3 - 3c_1)u_n^2v_n + (c_3 - c_2)v_{n-1}v_n + c_2u_{n-1}u_nv_{n-1} + (c_2 - c_3)v_n^2$$

$$+ \left[(c_3 - c_2)v_{n-1}v_n + c_2u_{n-1}u_nv_{n-1} + c_2v_{n-1}^2\right]$$

$$- \left[(c_3 - c_2)v_{n+1}v_n + c_2u_nv_{n+1}v_n + c_2v_n^2\right].$$

The terms inside the square brackets determine:

$$J_n = (c_3 - c_2)v_{n-1}v_n + c_2u_{n-1}u_nv_{n-1} + c_2v_{n-1}^2.$$
The terms outside the square brackets must vanish, thus
\[ S = \{3c_1 - c_2 = 0, c_3 - 3c_1 = 0, c_2 - c_3 = 0\}. \]
The solution is \(3c_1 = c_2 = c_3\), so choose \(c_1 = \frac{1}{3}\), and \(c_2 = c_3 = 1\):
\[
\rho_n = \frac{1}{3}u_n^3 + u_n(v_{n-1} + v_n), \quad J_n = u_{n-1}u_nv_{n-1} + v_{n-1}^2.
\]
Analogously, conserved densities of rank \(\leq 5\):

\[
\begin{align*}
\rho_n^{(1)} &= u_n & \rho_n^{(2)} &= \frac{1}{2}u_n^2 + v_n \\
\rho_n^{(3)} &= \frac{1}{3}u_n^3 + u_n(v_{n-1} + v_n) \\
\rho_n^{(4)} &= \frac{1}{4}u_n^4 + u_n^2(v_{n-1} + v_n) + u_nu_{n+1}v_n + \frac{1}{2}v_n^2 + v_nv_{n+1} \\
\rho_n^{(5)} &= \frac{1}{5}u_n^5 + u_n^3(v_{n-1} + v_n) + u_nu_{n+1}v_n(u_n + u_{n+1}) + u_nv_{n-1}(v_{n-2} + v_{n-1} + v_n) + u_nv_n(v_{n-1} + v_n + v_{n+1}).
\end{align*}
\]
• GENERALIZED SYMMETRIES

A vector function $G(\ldots, u_{n-1}, u_n, u_{n+1}, \ldots)$ is a symmetry if the infinitesimal transformation $u_n \rightarrow u_n + \epsilon G(\ldots, u_{n-1}, u_n, u_{n+1}, \ldots)$ leaves the lattice system invariant within order $\epsilon$. Consequently, $G$ must satisfy the linearized equation

$$D_t G = F'(u_n)[G],$$

where $F'$ is the Fréchet derivative of $F$, i.e.,

$$F'(u_n)[G] = \frac{\partial}{\partial \epsilon} F(u_n + \epsilon G)|_{\epsilon=0}.$$  

Here, $u_n \rightarrow u_n + \epsilon G(\ldots, u_{n-1}, u_n, u_{n+1}, \ldots)$ means that $u_{n+k}$ is replaced by $u_{n+k} + \epsilon G|_{n \rightarrow n+k}$.

• Example

Consider the Toda lattice

$$\dot{u}_n = v_{n-1} - v_n, \quad \dot{v}_n = v_n(u_n - u_{n+1}).$$

Higher-order symmetry of rank $(3, 4)$:

$$G_1 = v_n(u_n + u_{n+1}) - v_{n-1}(u_{n-1} + u_n),$$

$$G_2 = v_n(u_{n+1}^2 - u_n^2) + v_n(v_{n+1} - v_{n-1}).$$
• Algorithm for Generalized Symmetries of DDEs.

Consider the Toda system with $w(u_n) = 1$ and $w(v_n) = 2$. Compute the form of the symmetry of ranks $(3, 4)$, i.e. the first component of the symmetry has rank 3, the second rank 4.

**Step 1: Construct the form of the symmetry.**

List all monomials in $u_n$ and $v_n$ of rank 3 or less:

\[ \mathcal{L}_1 = \{ u_n^3, u_n^2, u_nv_n, u_n, v_n \}, \]

and of rank 4 or less:

\[ \mathcal{L}_2 = \{ u_n^4, u_n^3, u_n^2v_n, u_n^2, u_nv_n, u_n, v_n^2, v_n \}. \]

For each monomial in $\mathcal{L}_1$ and $\mathcal{L}_2$, introduce enough $t$-derivatives, so that each term exactly has rank 3 and 4, respectively.

Using the lattice equations, for the monomials in $\mathcal{L}_1$:

\[
\begin{align*}
\frac{d^0}{dt^0}(u_n^3) &= u_n^3, \\
\frac{d^0}{dt^0}(u_nv_n) &= u_nv_n, \\
\frac{d}{dt}(u_n^2) &= 2u_n\dot{u}_n = 2u_nv_{n-1} - 2u_nv_n, \\
\frac{d}{dt}(v_n) &= \dot{v}_n = u_nv_n - u_{n+1}v_n, \\
\frac{d^2}{dt^2}(u_n) &= \frac{d}{dt}(\dot{u}_n) = \frac{d}{dt}(v_{n-1} - v_n) \\
&= u_{n-1}v_{n-1} - u_nv_{n-1} - u_nv_n + u_{n+1}v_n.
\end{align*}
\]

Gather the resulting terms:

\[ \mathcal{R}_1 = \{ u_n^3, u_{n-1}v_{n-1}, u_nv_{n-1}, u_nv_n, u_{n+1}v_n \}. \]
\[ \mathcal{R}_2 = \{ u_n^4, u_{n-1}^2 v_{n-1}, u_{n-1} u_n v_{n-1}, u_n^2 v_{n-1}, v_n^2, u_n^2 v, u_n u_{n+1} v, u_{n+1}^2 v, u_{n+1} v_n, v_{n-1} v_n, v_n^2, v_n v_{n+1} \}. \]

Linearly combine the monomials in \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \)

\[
\begin{align*}
G_1 &= c_1 u_n^3 + c_2 u_{n-1} v_{n-1} + c_3 u_n v_{n-1} + c_4 u_n v_n + c_5 u_{n+1} v_n, \\
G_2 &= c_6 u_n^4 + c_7 u_{n-1}^2 v_{n-1} + c_8 u_{n-1} u_n v_{n-1} + c_9 u_n^2 v_{n-1} \\
&\quad + c_{10} v_{n-2} v_{n-1} + c_{11} v_{n-1}^2 + c_{12} u_n^2 v_n + c_{13} u_n u_{n+1} v_n \\
&\quad + c_{14} u_{n+1}^2 v_n + c_{15} v_{n-1} v_n + c_{16} v_n^2 + c_{17} v_n v_{n+1}.
\end{align*}
\]

**Step 2: Determine the unknown coefficients.**

Require that the symmetry condition holds.

Solution:

\[
\begin{align*}
c_1 &= c_6 = c_7 = c_8 = c_9 = c_{10} = c_{11} = c_{13} = c_{16} = 0, \\
-c_2 &= -c_3 = c_4 = c_5 = -c_{12} = c_{14} = -c_{15} = c_{17}.
\end{align*}
\]

Therefore, with \( c_{17} = 1 \), the symmetry of rank \((3, 4)\) is:

\[
\begin{align*}
G_1 &= u_n v_n - u_{n-1} v_{n-1} + u_{n+1} v_n - u_n v_{n-1}, \\
G_2 &= u_{n+1}^2 v_n - u_n^2 v_n + v_n v_{n+1} - v_{n-1} v_n.
\end{align*}
\]

Analogously, the symmetry of rank \((4, 5)\) reads

\[
\begin{align*}
G_1 &= u_n^2 v_n + u_n u_{n+1} v_n + u_{n+1}^2 v_n + v_n^2 + v_n v_{n+1} - u_{n-1}^2 v_{n-1} \\
&\quad - u_{n-1} u_n v_{n-1} - u_n^2 v_{n-1} - v_{n-2} v_{n-1} - v_{n-1}^2, \\
G_2 &= u_{n+1} v_n^2 + 2 u_{n+1} v_n v_{n+1} + u_{n+2} v_n v_{n+1} - u_n^3 v_n + u_{n+1}^3 v_n \\
&\quad - u_{n-1} v_{n-1} v_n - 2 u_n v_{n-1} v_n - u_n v_n^2.
\end{align*}
\]
• Example: Nonlinear Schrödinger (NLS) equation.

Ablowitz and Ladik discretization of the NLS equation:

\[ i \dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + u_n^* u_n (u_{n+1} + u_{n-1}) \]

\( u_n^* \) is the complex conjugate of \( u_n \).

Treat \( u_n \) and \( v_n = u_n^* \) as independent variables and add the complex conjugate equation. Absorb \( i \) in the scale on \( t \):

\[ \dot{u}_n = u_{n+1} - 2u_n + u_{n-1} + u_n v_n (u_{n+1} + u_{n-1}), \]
\[ \dot{v}_n = -(v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}). \]

Since \( v_n = u_n^* \), \( w(v_n) = w(u_n) \).

No uniformity in rank! Introduce an auxiliary parameter \( \alpha \) with weight.

\[ \dot{u}_n = \alpha (u_{n+1} - 2u_n + u_{n-1}) + u_n v_n (u_{n+1} + u_{n-1}), \]
\[ \dot{v}_n = -\alpha (v_{n+1} - 2v_n + v_{n-1}) - u_n v_n (v_{n+1} + v_{n-1}). \]

Uniformity in rank leads to

\[ w(u_n) + 1 = w(\alpha) + w(u_n) = 2w(u_n) + w(v_n) = 3w(u_n), \]
\[ w(v_n) + 1 = w(\alpha) + w(v_n) = 2w(v_n) + w(u_n) = 3w(v_n). \]

which yields

\[ w(u_n) = w(v_n) = \frac{1}{2}, w(\alpha) = 1. \]

Uniformity in rank is essential for steps 1 and 2.

After Step 2, set \( \alpha = 1 \). Step 3 leads to the result:

\[ \rho_n^{(1)} = c_1 u_n v_{n-1} + c_2 u_n v_{n+1}, \text{ etc.} \]
PART III: Software

*Scope and Limitations of Algorithms.*

- Systems of evolution equations or lattice equations must be polynomial in dependent variables.
  No *explicitly* dependencies on the independent variables.
- Only one space variable (continuous or discretized) is allowed.
- Program only computes polynomial conservation laws and generalized symmetries (no recursion operators yet).
- Program computes conservation laws and symmetries that explicitly depend on the independent variables, if the highest degree is specified.
- No limit on the number of equations in the system.
  In practice: time and memory constraints.
- Input systems may have (nonzero) parameters.
  Program computes the compatibility conditions for parameters such that conservation laws and symmetries (of a given rank) exist.
- Systems can also have parameters with (unknown) weight.
  This allows one to test evolution and lattice equations of non-uniform rank.
- For systems where one or more of the weights is free, the program prompts the user for info.
- Fractional weights and ranks are permitted.
- Complex dependent variables are allowed.
- PDEs and lattice equations must be of first-order in $t$. 
• Conclusions and Future Research

– Implement the recursion operator algorithm for PDEs.
– Design an algorithm for recursion operators of DDEs.
– Improve software, compare with other packages.
– Add tools for parameter analysis (Gröbner basis).
– Generalization towards broader classes of equations (e.g. $u_{xt}$).
– Generalization towards more space variables (e.g. Kadomtsev-Petviashvili equation).
– Conservation laws with time and space dependent coefficients.
– Conservation laws with $n$ dependent coefficients.
– Exploit other symmetries in the hope to find conserved densities of non-polynomial form
– Application: test models for integrability.
– Application: study of classes of nonlinear PDEs or DDEs.
– Compute constants of motion for dynamical systems (e.g. Lorenz and Hénon-Heiles systems)
• Implementation in Mathematica – Software


- Software: available via FTP, ftp site mines.edu in

  pub/papers/math_cs_dept/software/condens
  pub/papers/math_cs_dept/software/diffdens

  or via the Internet

  URL: http://www.mines.edu/fs_home/whereman/
• Publications


