SYMBOLIC COMPUTATION
OF
CONSERVED DENSITIES

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I. INTRODUCTION

Symbolic Software

- Solitons via Hirota’s method (Macsyma & Mathematica)
- Painlevé test for ODEs or PDEs (Macsyma)
- Conservation laws of PDEs (Mathematica)
- Lie symmetries for ODEs and PDEs (Macsyma)

Purpose of the programs

- Study of integrability of nonlinear PDEs
- Exact solutions as bench mark for numerical algorithms
- Classification of nonlinear PDEs
- Lie symmetries $\rightarrow$ solutions via reductions

Collaborators

- Ünal Göktaş (MS student)
- Chris Elmer (MS student)
- Wuning Zhuang (MS student)
- Ameina Nuseir (Ph.D student)
- Mark Coffey (CU-Boulder)
II. MATHEMATICA PROGRAM FOR CONSERVED DENSITIES

• Purpose

Compute polynomial-type conservation laws of single PDEs and systems of PDEs

Conservation law:
\[ \rho_t + J_x = 0 \]

both \( \rho(u, u_x, u_{2x}, \ldots, u_{nx}) \) and \( J(u, u_x, u_{2x}, \ldots, u_{nx}) \)

Consequently
\[ P = \int_{-\infty}^{+\infty} \rho dx = \text{constant} \]

provided \( J \) vanishes at infinity

Compare with constants of motions in classical mechanics
Example

Consider the KdV equation

\[ u_t + u u_x + u_{3x} = 0 \]

Conserved densities:

\[
\begin{align*}
\rho_1 &= u \\
\rho_2 &= u^2 \\
\rho_3 &= u^3 - 3u_x^2 \\
\vdots \\
\rho_6 &= u^6 - 60u^3u_x^2 - 30u_x^4 + 108u^2u_{2x}^2 \\
&+ \frac{720}{7}u_{2x}^3 - \frac{648}{7}uu_{3x}^2 + \frac{216}{7}u_{4x}^2 \\
\vdots
\end{align*}
\]

Integrable equations have \( \infty \) conservation laws
Algorithm and Implementation

Consider the scaling (weights) of the KdV

\[ u \sim \frac{\partial^2}{\partial x^2}, \quad \frac{\partial}{\partial t} \sim \frac{\partial^3}{\partial x^3} \]

Compute building blocks of \( \rho_3 \)

(i) Start with building block \( u^3 \)

Divide by \( u \) and differentiate twice \( (u^2)_{2x} \)

Produces the list of terms

\[ [u_x^2, uu_{2x}] \rightarrow [u_x^2] \]

Second list: remove terms that are total derivative
with respect to \( x \) or total derivative
up to terms earlier in the list

Divide by \( u^2 \) and differentiate twice \( (u)_{4x} \)

Produces the list: \( [u_{4x}] \rightarrow [ ] \)

[ ] is the empty list
Gather the terms:

$$\rho_3 = u^3 + c[1]u_x^2$$

where the constant $c_1$ must be determined

(ii) Compute $\frac{\partial \rho_3}{\partial t} = 3u^2u_t + 2c_1u_xu_{xt}$

Replace $u_t$ by $-(uu_x + u_{xxx})$ and $u_{xt}$ by $-(uu_x + u_{xxx})_x$

(iii) Integrate the result with respect to $x$

Carry out all integrations by parts

$$\frac{\partial \rho_3}{\partial t} = -\left[\frac{3}{4}u^4 + (c_1 - 3)uu_x^2 + 3u^2u_{xx} - c_1u_{xx}^2 + 2c_1u_xu_{xxx}\right]_x$$

$$-(c_1 + 3)u_x^3$$

The last non-integrable term must vanish

Thus, $c_1 = -3$

Result:

$$\rho_3 = u^3 - 3u_x^2$$

(iv) Expression $[...]$ yields

$$J_3 = \frac{3}{4}u^4 - 6uu_x^2 + 3u^2u_{xx} + 3u_{xx}^2 - 6u_xu_{xxx}$$
Computer building blocks of $\rho_6$

(i) Start with $u^6$

Divide by $u$ and differentiate twice

$(u^5)_{2x}$ produces the list of terms

$[u^3u^2_x, u^4u_{2x}] \rightarrow [u^3u^2_x]$  

Next, divide $u^6$ by $u^2$, and compute $(u^4)_{4x}$

Corresponding list:

$[u^4_x, uu^2_xu_{2x}, u^2u^2_{2x}, u^2u_xu_{3x}, u^3u_{4x}] \rightarrow [u^4_x, u^2u^2_{2x}]$

Proceed with $(\frac{u^6}{u^3})_{6x} = (u^3)_{6x}, (\frac{u^6}{u^4})_{8x} = (u^2)_{8x}$

and $(\frac{u^6}{u^5})_{10x} = (u)_{10x}$

Obtain the lists:

$[u^3_{2x}, u_xu_{2x}u_{3x}, uu^2_{3x}, u^2u^2_xu_{4x}, uu_xu_{4x}, uu_xu_{5x}, u^2u_{6x}] \rightarrow [u^3_{2x}, uu^2_{3x}]$

$[u^2_{4x}, u_3xu_{5x}, u_2xu_{6x}, u_xu_{7x}, uu_{8x}] \rightarrow [u^2_{4x}]$

and $[u_{10x}] \rightarrow [\ ]$
Gather the terms:

\[ \rho_6 = u^6 + c_1 u^3 u_x^2 + c_2 u_x^4 + c_3 u^2 u_{2x}^2 + c_4 u_{2x}^3 + c_5 u u_{3x}^2 + c_6 u_{4x}^2 \]

where the constants \( c_i \) must be determined

(ii) Compute \( \frac{\partial}{\partial t} \rho_6 \)

Replace \( u_t, u_{xt}, \ldots, u_{nx,t} \) by \(- (u u_x + u_{xxx}), \ldots\)

(iii) Integrate the result with respect to \( x \)

Carry out all integrations by parts

Require that non-integrable part vanishes

Set to zero all the coefficients of the independent combinations involving powers of \( u \) and its derivatives with respect to \( x \)

Solve the linear system for unknowns \( c_1, c_2, \ldots, c_6 \)

Result:

\[ \rho_6 = u^6 - 60u^3 u_x^2 - 30u_x^4 + 108u^2 u_{2x}^2 + \frac{720}{7} u_{2x}^3 - \frac{648}{7} u u_{3x}^2 + \frac{216}{7} u_{4x}^2 \]

(iv) Flux \( J_6 \) can be computed by substituting the constants into the integrable part of \( \rho_6 \)
• Further Examples

* Conservation laws of generalized Schamel equation

\[ n^2 u_t + (n + 1)(n + 2)u^{2n}u_x + u_{xxx} = 0 \]

\( n \) positive integer

\[
\begin{align*}
\rho_1 &= u \\
\rho_2 &= u^2 \\
\rho_3 &= \frac{1}{2}u_x - \frac{n^2}{2}u^{2+\frac{2}{n}}
\end{align*}
\]

no further conservation laws

* Conserved densities of modified vector derivative nonlinear Schrödinger equation

\[
\frac{\partial B_{\perp}}{\partial t} + \frac{\partial}{\partial x}(B^2_{\perp}B_{\perp}) + \alpha B_{\perp0}B_{\perp0} \cdot \frac{\partial B_{\perp}}{\partial x} + e_x \times \frac{\partial^2 B_{\perp}}{\partial x^2} = 0
\]

Replace vector equation by

\[
\begin{align*}
u_t + (u(u^2 + v^2) + \beta u - v_x)_x &= 0 \\
v_t + (v(u^2 + v^2) + u_x)_x &= 0
\end{align*}
\]

\( u \) and \( v \) denote the components of \( B_{\perp} \) parallel and perpendicular to \( B_{\perp0} \) and \( \beta = \alpha B^2_{\perp0} \).
The first 5 conserved densities are:

\[ \rho_1 = u^2 + v^2 \]

\[ \rho_2 = \frac{1}{2}(u^2 + v^2)^2 - uv_x + u_x v + \beta u^2 \]

\[ \rho_3 = \frac{1}{4}(u^2 + v^2)^3 + \frac{1}{2}(u_x^2 + v_x^2) - u^3 v_x + v^3 u_x + \frac{\beta}{4}(u^4 - v^4) \]

\[ \rho_4 = \frac{1}{4}(u^2 + v^2)^4 - \frac{2}{5}(u_x v_{xx} - u_{xx} v_x) + \frac{4}{5}(u u_x + v v_x)^2 \]

\[ \quad + \frac{6}{5}(u^2 + v^2)(u_x^2 + v_x^2) - (u^2 + v^2)^2(u v_x - u_x v) \]

\[ \quad + \frac{\beta}{5}(2u_x^2 - 4u^3 v_x + 2u^6 + 3u^4 v^2 - v^6) + \frac{\beta^2}{5}u^4 \]
\[
\rho_5 = \frac{7}{16}(u^2 + v^2)^5 + \frac{1}{2}(u_{xx}^2 + v_{xx}^2)
\]
\[
- \frac{5}{2}(u^2 + v^2)(u_x v_{xx} - u_{xx} v_x) + 5(u^2 + v^2)(u u_x + v v_x)^2
\]
\[
+ \frac{15}{4}(u^2 + v^2)^2(u_x^2 + v_x^2)^2 - \frac{35}{16}(u^2 + v^2)^3(u v_x - u_x v)
\]
\[
+ \frac{\beta}{8}(5u^8 + 10u^6v^2 - 10u^2v^6 - 5v^8 + 20u^2u_x^2
\]
\[
- 12u^5v_x + 60uv^4v_x - 20v^2v_x^2)
\]
\[
+ \frac{\beta^2}{4}(u^6 + v^6)
\]
• Coupled Systems

* Conserved densities for the Coupled KdV Equations (Hirota-Satsuma system)

\[ u_t - a(u_{xxx} + 6uu_x) - 2bv v_x = 0 \]
\[ v_t + v_{xxx} + 3uv_x = 0 \]

\[ \rho_1 = u \]
\[ \rho_2 = u^2 + \frac{2}{3} bv^2 \]
\[ \rho_3 = (1 + a)(u^3 - \frac{1}{2}u^2_x) + b( u v^2 - v^2_x) \]

and e.g.

\[ \rho_4 = u^4 - 2uu^2_x + \frac{1}{5}u^2_{xx} \]
\[ + \frac{4}{5}b(u^2v^2 + \frac{2}{3}uvv_{xx} + \frac{8}{3}uv^2_x - \frac{2}{3}v^2_{xx}) + \frac{4}{15}b^2v^4 \]

provided \( a = \frac{1}{2} \)

There are infinitely many more conservation laws
* Conserved densities for the Ito system

\[ u_t - u_{xxx} - 6uu_x - 2vv_x = 0 \]
\[ v_t - 2u_xv - 2uv_x = 0 \]

\[ \rho_1 = \frac{1}{2}u \]
\[ \rho_2 = \frac{1}{2}(u^2 + v^2) \]
\[ \rho_3 = u^3 - \frac{1}{2}u_x^2 + uv^2 \]

and infinitely many more conservation laws
A Class of Fifth-order Evolution Equations

\[ u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma uu_{3x} + u_{5x} = 0 \]

Special cases:

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>Equation</th>
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</thead>
<tbody>
<tr>
<td>30</td>
<td>20</td>
<td>10</td>
<td>Lax</td>
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<td>5</td>
<td>5</td>
<td>5</td>
<td>Sawada Kotera or Caudry–Dodd–Gibbon</td>
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<td>20</td>
<td>25</td>
<td>10</td>
<td>Kaup–Kuperschmidt</td>
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<tr>
<td>2</td>
<td>6</td>
<td>3</td>
<td>Ito</td>
</tr>
<tr>
<td>Density</td>
<td>Sawada-Kotera equation</td>
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<tr>
<td>$\rho_1$</td>
<td>$u$</td>
<td>$u$</td>
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<tr>
<td>$\rho_2$</td>
<td>----</td>
<td>$\frac{1}{2}u^2$</td>
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<tr>
<td>$\rho_3$</td>
<td>$\frac{1}{3}u^3 - u_x^2$</td>
<td>$\frac{1}{3}u^3 - \frac{1}{6}u_x^2$</td>
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<tr>
<td>$\rho_4$</td>
<td>$\frac{1}{4}u^4 - \frac{9}{4}u_x^2 + \frac{3}{4}u_{2x}^2$</td>
<td>$\frac{1}{4}u^4 - \frac{1}{3}u_x^2 + \frac{1}{20}u_{2x}^2$</td>
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<tr>
<td>$\rho_6$</td>
<td>----</td>
<td>$\frac{1}{5}u^5 - u_x^2 + \frac{1}{5}u_{2x}^2 - \frac{1}{70}u_{3x}^2$</td>
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<tr>
<td>$\rho_6$</td>
<td>$\frac{1}{6}u^6 - \frac{25}{4}u_x^3 + \frac{17}{8}u_x^4 + 6u_{2x}^2 + 2u_x^3 - \frac{21}{8}u_xu_{3x}^2 + \frac{3}{8}u_{4x}^2$</td>
<td>$\frac{1}{6}u^6 - \frac{1}{3}u_x^3 + \frac{5}{36}u_x^4 + \frac{1}{2}u_{2x}^2$</td>
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<tr>
<td>$\rho_7$</td>
<td>$\frac{1}{7}u^7 - 9u_x^4u_x^2 - \frac{54}{5}u_xu_x^4 + \frac{57}{5}u_x^3u_{2x}^2$</td>
<td>$\frac{1}{7}u^7 - \frac{5}{3}u_x^4u_x^2 - \frac{5}{36}u_x^4 + \frac{1}{2}u_{2x}^2$</td>
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<td></td>
<td>$+ \frac{648}{35}u_x^2u_{2x}^2 + \frac{489}{35}u_x^3u_{3x}^2 - \frac{261}{35}u_x^2u_{3x}^2$</td>
<td>$+ \frac{1}{2}u_x^2u_{2x}^2 + \frac{10}{21}u_xu_{3x}^2 - \frac{3}{14}u_{2x}^2$</td>
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<td></td>
<td>$- \frac{288}{35}u_x^2u_{3x}^2 + \frac{81}{35}u_xu_{4x}^2 - \frac{9}{35}u_{5x}^2$</td>
<td>$- \frac{5}{42}u_xu_{3x}^2 + \frac{1}{21}u_{2x}u_{4x}^2 - \frac{1}{942}u_{5x}^2$</td>
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<td>$\rho_8$</td>
<td>----</td>
<td>$\frac{1}{8}u^8 - \frac{7}{2}u_x^5u_x^2 - \frac{35}{12}u_x^2u_{2x}^4 + \frac{7}{4}u_{4x}^2$</td>
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<td>$+ \frac{7}{3}u_x^2u_{2x}^2 + \frac{5}{3}u_x^3u_{2x}^2 + \frac{7}{24}u_{2x}^4 + \frac{1}{2}u_{3x}^2$</td>
<td>$+ \frac{7}{3}u_x^2u_{2x}^2 - \frac{5}{6}u_{2x}u_{3x}^2 + \frac{1}{12}u_{2x}^2$</td>
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<td>$- \frac{1}{3}u_x^2u_{3x}^2 - \frac{5}{6}u_xu_{4x}^2 + \frac{1}{12}u_{2x}^2$</td>
<td>$+ \frac{7}{15}u_xu_{4x}^2 - \frac{1}{15}u_{3x}^2 + \frac{1}{34}u_{6x}^2$</td>
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<td>$\rho_4$</td>
<td>$\frac{u^4}{4} - \frac{9}{20} u u_x^2 + \frac{3}{64} u_x^4$</td>
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<tr>
<td>$\rho_5$</td>
<td>$\frac{u^5}{5} - \frac{27}{8} u^4 u_x - \frac{369}{320} u u_x^4 + \frac{69}{40} u_y^3 u_x^2$</td>
<td>$\frac{u^5}{5} - \frac{27}{8} u^4 u_x - \frac{369}{320} u u_x^4 + \frac{69}{40} u_y^3 u_x^2$</td>
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<td>$\rho_6$</td>
<td>$\frac{u^6}{6} - \frac{35}{16} u^3 u_x^2 - \frac{31}{256} u_x^4 + \frac{51}{64} u_x^2 u_x^2$</td>
<td>$\frac{u^6}{6} - \frac{35}{16} u^3 u_x^2 - \frac{31}{256} u_x^4 + \frac{51}{64} u_x^2 u_x^2$</td>
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<td>$\frac{u^7}{7} - \frac{27}{8} u^4 u_x + \frac{369}{320} u u_x^4 + \frac{69}{40} u^3 u_x^2$</td>
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