THE APPLICATION OF TWO DIMENSIONAL CYLINDRICAL
FINITE DIFFERENCE TIME DOMAIN METHOD TO
WAVEGUIDES AND SCATTERING
PROBLEMS

BY

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ABSTRACT

The Application of Two Dimensional Cylindrical Finite Difference Time Domain Method to Waveguides and Scattering Problems

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M.S., University of Mississippi, 1996, Thesis Directed by Dr. Atef Z. Elsherbeni and Dr. Charles E. Smith

The Finite Difference Time Domain (FDTD) method has been widely applied to solve electromagnetic problems, and for most cases, the cartesian coordinate system is applied in these FDTD applications. However, when modeling circular geometries, not only is the accuracy of the traditional staircasing FDTD method decreased, but the required computational space is also increased. Nevertheless, there are few publications dealing with the cylindrical coordinate FDTD (CC-FDTD) method. In this thesis, the formulation for a two dimensional cylindrical coordinate FDTD formulation for both scattering and waveguide type of problems is provided along with numerical examples for scattered fields and for the TE and TM mode cutoff frequencies inside a coaxial waveguide. To validate the formulation, cutoff frequencies inside coaxial waveguides are computed by using a developed CC-FDTD code, and the results are compared with analytical and numerical solutions obtained with the traditional staircasing FDTD method. The deviations in cutoff frequencies as compared to analytic solutions for the CC-FDTD methods are rather small as compared to the deviations for the traditional staircasing FDTD. In addition, the required computational space using CC-FDTD is reduced by up to one seventh of that required by the staircasing FDTD. Better accuracy and computer resource efficiency are also provided as CC-FDTD is applied to the scattering problem of a PEC cylinder with respect to the staircasing FDTD. To minimize the numerical dispersion from the CC-FDTD outer radiation boundary, three types of absorbing boundaries commonly used in cartesian coordinates are implemented in CC-FDTD and evaluated. It is concluded that for modeling geometries with circular boundaries, the cylindrical coordinate FDTD technique can not only provide better accuracy but also a more efficient computational procedure than traditional FDTD methods.
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<tr>
<td>c</td>
<td>Speed of light</td>
<td>(m/s)</td>
</tr>
<tr>
<td>d</td>
<td>Spatial distance</td>
<td>(m)</td>
</tr>
<tr>
<td>(\vec{E})</td>
<td>Electric field vector</td>
<td>(V/m)</td>
</tr>
<tr>
<td>(\vec{H})</td>
<td>Magnetic field vector</td>
<td>(A/m)</td>
</tr>
<tr>
<td>(J_z)</td>
<td>Electric current densities</td>
<td>(A/m)</td>
</tr>
<tr>
<td>(\hat{k})</td>
<td>Unit incident wave vector</td>
<td>(Dimensionless)</td>
</tr>
<tr>
<td>(M_z)</td>
<td>Magnetic current densities</td>
<td>(A/m)</td>
</tr>
<tr>
<td>(N_\rho)</td>
<td>Number of cells in (\rho) direction</td>
<td>(Dimensionless)</td>
</tr>
<tr>
<td>(N_\phi)</td>
<td>Number of cells in (\phi) direction</td>
<td>(Dimensionless)</td>
</tr>
<tr>
<td>(\rho, \phi)</td>
<td>Cylindrical coordinate</td>
<td>(m), (Degree)</td>
</tr>
<tr>
<td>(\phi_i)</td>
<td>Angle of incident wave measured from (x) axis</td>
<td>(Degree)</td>
</tr>
<tr>
<td>(\Delta\rho)</td>
<td>Cell length in (\rho) direction</td>
<td>(m)</td>
</tr>
<tr>
<td>(\Delta\phi)</td>
<td>Cell angle</td>
<td>(Degree)</td>
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<tr>
<td>(R(\theta))</td>
<td>Reflection factor</td>
<td>(Dimensionless)</td>
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<tr>
<td>(\sigma^*)</td>
<td>Magnetic conductivity</td>
<td>(S/m)</td>
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<tr>
<td>(\sigma)</td>
<td>Electrical conductivity</td>
<td>(S/m)</td>
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<tr>
<td>(t_d)</td>
<td>Time delay parameter</td>
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CHAPTER I

INTRODUCTION

Since Maxwell's differential equations were formulated in about 1870, scientists and engineers have tried to find solutions to these equations, which can predict the behavior of electromagnetic wave propagation [1]. The finite difference technique is one of the numerical methods providing effective solutions to Maxwell's differential equations that has been widely used recently in many numerical applications. There are two categories of finite difference methods used for solving electromagnetic problems. One approach is to solve Maxwell's differential equation in the frequency domain, the other approach is to obtain the solution in the time domain. Based on the progress of computer techniques and limitations of frequency domain solutions, finite difference techniques in the time domain have become a valuable solution technique [1]. By using the fast Fourier transform (FFT) to transform time domain finite difference results, not only can solutions be provided for frequency domain, but these results can provide insights into wave behavior in time domain.

With Yee's definition of a basic cell, the foundation of the finite difference time domain (FDTD) method based on the field spatial arrangements defined in cartesian coordinates and in the time domain, was presented in 1966 [2]. Since that time applications of FDTD techniques to solve electromagnetic problems have been successfully applied for several years, because of the efficiency and accuracy of these methods. Cartesian coordinates are used in most FDTD applications. However, when modeling circular or coaxial geometries, the accuracy and computational space efficiency
are limited by using staircasing cartesian FDTD methods. Although there are several techniques that have been adopted in cartesian coordinates to improve staircasing FDTD methods [3-4], the publications dealing with applications of cylindrical cells to model circular geometries are very few [5-7,13]. Therefore, there is a need to study a cylindrical coordinate FDTD method (CC-FDTD).

The purpose of this study is to develop a two dimensional cylindrical coordinate FDTD formulation for both scattering and waveguide types of problems and the associated absorbing boundary conditions (ABCs). To validate the formulation, programs are developed and tested using the FORTRAN language. Analytical solutions and computational results using traditional staircasing FDTD methods, numerical scattered field examples of circular scatterers excited by z-directed line sources as well as TM\textsuperscript{z} and TE\textsuperscript{z} normally incident plane waves, and both TE\textsuperscript{z} and TM\textsuperscript{z} mode cutoff frequencies inside coaxial waveguides are presented in this thesis to demonstrate the accuracy and efficiency of this method. Better accuracy and computer resource efficiency of computational results have been obtained by the cylindrical FDTD technique than the staircasing FDTD. Furthermore, various absorbing boundary conditions have also been adopted and compared in cylindrical coordinates. In order to determine an optimal absorbing boundary condition for cylindrical FDTD methods, a numerical scattering example is computed by the cylindrical FDTD with various absorbing boundary conditions using the same computational space. Using the analytical solutions as the standard, global errors caused by different absorbing boundary conditions are compared and analyzed to evaluate these absorbing boundary conditions in cylindrical coordinates. This study not only provides a better method for modeling circular or coaxial geometries but also led to a better
understanding of the non-uniform cell applications in finite difference time domain techniques.
CHAPTER II

FORMULATION FOR THE TWO DIMENSIONAL SCATTERING

CC-FDTD METHOD

2.1 Introduction

Although FDTD methods have been applied and studied for almost 30 years, most applications are based on the finite difference approximations to Maxwell’s differential equations in Cartesian coordinates. Since the vector operations in cylindrical and rectangular coordinates are different, the finite difference approximation to Maxwell’s differential equations in cylindrical coordinates are also different. Therefore, there is a need to develop the formulation for cylindrical FDTD.

Since two dimensional waveguides and scattering problems are studied in this thesis, the derivation of the basic formulas have assumed invariance with respect to the z axis. In FDTD techniques, there are four major parts: the field updating equations, the time step size, the source type, and the absorbing boundary conditions. Cylindrical vector operations are applied in Maxwell’s differential equations to derive the formulation for these four parts of the cylindrical FDTD. These four major components will be introduced in the following four sections.

2.2 Derivation of Field Updating Equations

The updating field equations are the kernel of any FDTD program, and each updating equation should satisfy Maxwell’s equations and follow the field spatial arrangement as described by a Yee cell [7].
The following two Maxwell’s equations [8],

\[
\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E} + \mathbf{J}_i \tag{1.a}
\]

\[
\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} - \sigma^* \mathbf{H} - \mathbf{M}_i \tag{1.b}
\]

are used to develop the updating equations for \( \mathbf{E} \) and \( \mathbf{H} \). Since two dimensional \( z \)-invariant scattering problems are concerned, the field updating equations in the following subsections are assumed without \( z \)-variation. In other words, the derivative of each field with respect to the \( z \)-direction is zero. The \( \mathbf{J}_i \) and \( \mathbf{M}_i \) are the electric and magnetic sources in the medium.

### 2.2.1 Field Updating Equations for Near Zone Sources

According to equation (1.a) along with the time and spatial relations of each field component in a cylindrical coordinate system as shown in Fig. 1, the general \( E \)-field updating equations can be expressed as follows,

\[
\mathbf{E}^n(I, J) = \frac{\varepsilon}{(\varepsilon + \sigma \Delta t)} \left[ \mathbf{E}^{n-1}(I, J) - \frac{\Delta t}{\varepsilon} \mathbf{J}_i^n(I, J) + \frac{\Delta t}{\varepsilon} \nabla \times \mathbf{H}^{n-\frac{1}{2}}(I, J) \right] \tag{2}
\]

In cylindrical coordinates, the vector differential operation is expressed as

\[
\nabla \times \mathbf{A} = \hat{\rho} \left( \frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\rho}{\partial z} \right) + \hat{\phi} \left( \frac{\partial A_\rho}{\partial z} - \frac{\partial A_\phi}{\partial \rho} \right) + \hat{z} \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho A_\phi \right) - \frac{1}{\rho} \frac{\partial A_\rho}{\partial \phi} \right) \tag{3}
\]
Fig. 1 Placements of the field components in a two-dimensional cylindrical coordinate Yee cell.
Using the vector differential operation with the E-field general updating equation (2), the E-field components can be obtained as

\[
E_x^n (I, J) = \left[ \frac{\varepsilon}{\varepsilon + \sigma \Delta t} \right] E_x^{n-1} (I, J) + \frac{\Delta t}{\varepsilon \left( I + \frac{1}{2} \right) \Delta \rho} \left[ \frac{H_z^{n-\frac{1}{2}} (I, J) - H_z^{n-\frac{1}{2}} (I, J - 1)}{\Delta \phi} \right] - \frac{\Delta t}{\varepsilon} J_{\rho x} (I, J) \]  

\text{(4.a)}

\[
E_y^n (I, J) = \left[ \frac{\varepsilon}{\varepsilon + \sigma \Delta t} \right] E_y^{n-1} (I, J) - \frac{\Delta t}{\varepsilon} \left[ \frac{H_z^{n-\frac{1}{2}} (I, J) - H_z^{n-\frac{1}{2}} (I - 1, J)}{\Delta \rho} \right] - \frac{\Delta t}{\varepsilon} J_{\rho y} (I, J) \]  

\text{(4.b)}

\[
E_z^n (I, J) = \left[ \frac{\varepsilon}{\varepsilon + \sigma \Delta t} \right] E_z^{n-1} (I, J) + \frac{\Delta t}{\varepsilon (I + \frac{1}{2}) \Delta \rho} \left[ \frac{I + \frac{1}{2}}{2} \Delta \rho H_{\rho z}^{n-\frac{1}{2}} (I, J) - \frac{I - \frac{1}{2}}{2} \Delta \rho H_{\rho z}^{n-\frac{1}{2}} (I - 1, J) \right] - \frac{\Delta t}{\varepsilon} J_{\rho z} (I, J) \]  

\text{(4.c)}

The above updating equations (4.a-c) are for two dimensional, z-invariant, near zone electric sources. Similarly, the general H-field updating equations can be obtained.
by using the same procedure applied in the E-field updating equations. The resulting general updating equation for H field components can be expressed as

\[
H_{n+\frac{1}{2}}^{\mu}(I,J) = \frac{\mu}{(\mu + \sigma^{*} \Delta t)} \left[ H_{n-\frac{1}{2}}^{\mu}(I,J) - \frac{\Delta t}{\mu} M_{i}^{n-\frac{1}{2}}(I,J) - \frac{\Delta t}{\mu} \nabla \times E_{n}(I,J) \right]
\]

(5)

If the vector differential operation is applied to (5), the H-field components can be written as

\[
H_{\rho}^{n+\frac{1}{2}}(I,J) = \frac{\mu}{(\mu + \sigma^{*} \Delta t)} \left[ H_{\rho}^{n-\frac{1}{2}}(I,J) - \frac{\Delta t}{\mu(\Delta \rho)} \left[ \frac{E_{\rho}(I,J+1) - E_{\rho}(I,J)}{\Delta \rho} \right] \right]
\]

\[- \frac{\Delta t}{\mu} M_{\rho}^{n+\frac{1}{2}}(I,J) \]

(6.a)

\[
H_{\phi}^{n+\frac{1}{2}}(I,J) = \frac{\mu}{(\mu + \sigma^{*} \Delta t)} \left[ H_{\phi}^{n-\frac{1}{2}}(I,J) + \frac{\Delta t}{\mu} \left[ \frac{E_{\phi}(I+1,J) - E_{\phi}(I,J)}{\Delta \rho} \right] \right]
\]

\[- \frac{\Delta t}{\mu} M_{\phi}^{n+\frac{1}{2}}(I,J) \]

(6.b)
\[ H_I^{n+\frac{1}{2}}(I, J) = \frac{\mu}{(\mu + \sigma^* \Delta t)} \left[ H_I^{n+\frac{1}{2}}(I, J) - \frac{\Delta t}{\mu (I + \frac{1}{2}) \Delta \rho} \right. \]
\[ \left. \left[ \left( I + \frac{1}{2} \right) \Delta \rho E_{\phi}^n(I + 1, J) - \left( I - \frac{1}{2} \right) \Delta \rho E_{\phi}^n(I, J) \right] + \frac{\Delta t}{\mu} M_{I, i}^{n+\frac{1}{2}}(I, J) \right] \]

(6.c)

2.2.2 Field Updating Equations for Far-Zone Sources

When the excitation is based on radiation from sources located in the far field of the scattering objects (resulting in a plane wave type of excitation in the near zone in most cases), Maxwell’s equations with \( J_i = 0 \), can be written as

\[ \nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E} \]  

(7.a)

where \( \mathbf{E} \) and \( \mathbf{H} \) are total electrical and magnetic fields respectively, which can be separated into incident and scattering fields, i.e.

\[ \mathbf{E} = \mathbf{E}' = \mathbf{E}^i + \mathbf{E}^s \]  

and  

\[ \mathbf{H} = \mathbf{H}' = \mathbf{H}^i + \mathbf{H}^s \]  

(7.b)

where the magnetic incident field can be related to the incident electric field by

\[ \nabla \times \mathbf{H}' = \varepsilon_0 \frac{\partial \mathbf{E}'}{\partial t} \]  

(7.c)
Substituting (7.b-c) into (7.a), the corresponding electrical field general updating equation can be derived in the form,

\[
\mathbf{E}^{i,n}(I,J) = -\mathbf{E}^{i,n}(I,J) + \frac{\varepsilon}{(\varepsilon + \sigma \Delta t)} \left[ \mathbf{E}^{s,n-1}(I,J) + \frac{\varepsilon_o}{\varepsilon} \mathbf{E}^{i,n}(I,J) + \frac{\varepsilon - \varepsilon_o}{\varepsilon} \mathbf{E}^{i,n-1}(I,J) \right] \\
+ \frac{\Delta t}{\varepsilon} \nabla \times \mathbf{H}^{s,n-\frac{1}{2}} + \frac{\Delta t}{\varepsilon} \nabla \times \mathbf{H}^{s,n-\frac{1}{2}}
\]  

(8)

According to equation (8) and (3), the components of the electric field in cylindrical coordinates are

\[
\mathbf{E}_\rho^{s,n}(I,J) = -\mathbf{E}_\rho^{i,n}(I,J) + \frac{\varepsilon}{(\varepsilon + \sigma \Delta t)} \left[ \mathbf{E}_\rho^{s,n-1}(I,J) + \frac{\varepsilon_o}{\varepsilon} \mathbf{E}_\rho^{i,n}(I,J) + \frac{\varepsilon - \varepsilon_o}{\varepsilon} \mathbf{E}_\rho^{i,n-1}(I,J) \right] \\
+ \frac{\Delta t}{\varepsilon} \frac{\mathbf{H}^{s,n-\frac{1}{2}}(I,J) - \mathbf{H}^{s,n-\frac{1}{2}}(I,J-1)}{\Delta \phi}
\]

(9.a)

\[
\mathbf{E}_\phi^{s,n}(I,J) = -\mathbf{E}_\phi^{i,n}(I,J) + \frac{\varepsilon}{(\varepsilon + \sigma \Delta t)} \left[ \mathbf{E}_\phi^{s,n-1}(I,J) + \frac{\varepsilon_o}{\varepsilon} \mathbf{E}_\phi^{i,n}(I,J) + \frac{\varepsilon - \varepsilon_o}{\varepsilon} \mathbf{E}_\phi^{i,n-1}(I,J) \right] \\
- \frac{\Delta t}{\varepsilon} \frac{\mathbf{H}^{s,n-\frac{1}{2}}(I,J) - \mathbf{H}^{s,n-\frac{1}{2}}(I,J-1)}{\Delta \rho}
\]

(9.b)
\[
E_{z,n}^{s,n}(I,J) = -E_{z,n}^{i,n}(I,J) + \frac{\varepsilon}{(\varepsilon + \sigma \Delta t)} \left[ E_{z,n-1}^{s,n-1}(I,J) + \frac{\varepsilon_0}{\varepsilon} E_{z,n-1}^{s,n-1}(I,J) + \frac{\varepsilon - \varepsilon_0}{\varepsilon} E_{z,n-1}^{i,n-1}(I,J) \right] \\
+ \frac{\Delta t}{\varepsilon (I \Delta \rho)} \left[ \frac{(I + \frac{1}{2}) \Delta \rho H_{\phi}^{s,n-1/2}(I,J) - (I - \frac{1}{2}) \Delta \rho H_{\phi}^{s,n-1/2}(I-1,J)}{\Delta \rho} \right] - \frac{H_{\rho}^{s,n-1/2}(I,J) - H_{\rho}^{s,n-1/2}(I,J-1)}{\Delta \phi} \right] 
\] (9.c)

Using a method similar to the development of the electrical field components where one starts from Maxwell equations with \( M_i \) equal to zero, we get

\[
\nabla \times E = -\varepsilon \frac{\partial H}{\partial t} - \sigma^* \mathbf{H} \quad (10.a)
\]

and with the assumption in equations (7.b) and (7.c), the electrical incident field can be expressed as

\[
\nabla \times E' = -\mu_0 \frac{\partial H^i}{\partial t} \quad (10.b)
\]

The magnetic field general updating equation can be expressed as
\[ \mathbf{H}^{s,n+\frac{1}{2}}(I,J) = -\mathbf{H}^{s,n+\frac{1}{2}}(I,J) + \frac{\mu}{\mu + \sigma \Delta t} \left[ \mathbf{H}^{s,n+\frac{1}{2}}(I,J) + \frac{\mu_0}{\mu} \mathbf{H}^{s,n+\frac{1}{2}}(I,J) \right]. \]

\[ + \left( \frac{\mu - \mu_0}{\mu} \right) \mathbf{H}^{i,n+\frac{1}{2}} - \frac{\Delta t}{\mu} \nabla \times \mathbf{E}^{s,n} \]

(11)

In cylindrical coordinates, the magnetic field updating equations can be expressed as

\[ \mathbf{H}_{\rho}^{s,n+\frac{1}{2}}(I,J) = -\mathbf{H}_{\rho}^{s,n+\frac{1}{2}}(I,J) + \frac{\mu}{\mu + \sigma \Delta t} \left[ \mathbf{H}_{\rho}^{s,n+\frac{1}{2}}(I,J) + \frac{\mu_0}{\mu} \mathbf{H}_{\rho}^{s,n+\frac{1}{2}}(I,J) \right]. \]

\[ - \frac{\mu - \mu_0}{\mu} \mathbf{H}_{\rho}^{i,n+\frac{1}{2}}(I,J) - \frac{\Delta t}{\mu} \left[ \frac{E_{2}^{z,n}(I,J+1) - E_{2}^{z,n}(I,J)}{\Delta \phi} \right]. \]

(12.a)

\[ \mathbf{H}_{\phi}^{s,n+\frac{1}{2}}(I,J) = -\mathbf{H}_{\phi}^{s,n+\frac{1}{2}}(I,J) + \frac{\mu}{\mu + \sigma \Delta t} \left[ \mathbf{H}_{\phi}^{s,n+\frac{1}{2}}(I,J) + \frac{\mu_0}{\mu} \mathbf{H}_{\phi}^{s,n+\frac{1}{2}}(I,J) \right]. \]

\[ + \frac{\mu - \mu_0}{\mu} \mathbf{H}_{\phi}^{i,n+\frac{1}{2}}(I,J) + \frac{\Delta t}{\mu} \left[ \frac{E_{2}^{z,n}(I,J+1) - E_{2}^{z,n}(I,J)}{\Delta \rho} \right]. \]

(12.b)
\[
H_z^{s,n+\frac{1}{2}}(I,J) = -H_z^{s,n+\frac{1}{2}}(I,J) + \frac{\mu}{(\mu + \sigma \Delta t)} \left[ H_z^{s,n-\frac{1}{2}}(I,J) + \frac{\mu_\phi}{\mu} H_z^{s,n+\frac{1}{2}}(I,J) \right]
\]
\[
+ \frac{\mu - \mu_\phi}{\mu} H_z^{s,n-\frac{1}{2}}(I,J) - \frac{\Delta t}{\mu (I+\frac{1}{2}) \Delta \rho}
\]
\[
\left[ \frac{(I + \frac{1}{2}) \Delta \rho E_\phi^{s,n}(I+1,J) - (I - \frac{1}{2}) \Delta \rho E_\phi^{s,n}(I,J)}{\Delta \rho} - \frac{E_\rho^{s,n}(I,J+1) - E_\rho^{s,n}(I,J)}{\Delta \phi} \right]
\]

(12.c)

By using equations (4) and (6) for two dimensional near zone sources or by using equations (9) and (12) for two dimensional far zone sources, the field updating subroutines for two dimensional CC-FDTD method can be developed.

2.3 Determination of Time Step Size

The determination of the time step in this method is another important factor that affects the stability of the FDTD technique [7]. In one time step, any point on the wave must not pass through more than one cell [7], which is known as the Courant limit. In the cylindrical coordinate system this is related to a cell where one side of the cell is \( l_\phi = \rho \Delta \phi \) and the other side is \( \Delta \rho \) as shown in Fig. 1. In other words, one cell side (\( l_\phi \)) is variant with the radial position \( \rho \). The cylindrical coordinates can be also treated as a special category of nonuniform orthogonal grids. Therefore, the time step for the problem can be expressed by considering the side length of the innermost cell which has the smallest cell dimensions. For this case, the Courant limit implies that
\[ \Delta t \leq \frac{1}{c \sqrt{\frac{1}{(\Delta \rho)^2} + \frac{1}{(t_{\phi_{\text{min}}})^2}}} \]  \hspace{1cm} (13)

which leads to a very small value of \( \Delta t \) with respect to the outermost cells.

2.4 Excitations in the Updating Equations

2.4.1 Categories of Source Functions

Different types of excitation functions can be employed in the updating equations. One of the simplest ways is to assign an initial value of a field component for one or several time steps that functions as a current source with a rectangular waveform and also provides very wide frequency domain information.

A Gaussian pulse is one of most widely adopted excitations because of the smooth transition at both the leading and trailing edges of the pulse. The Gaussian pulse can be expressed as [9]

\[ G(\rho, \phi, n\Delta t) = \exp \left[ -\frac{(n\Delta t - t_{o} - t_{d})^2}{T^2} \right] \]  \hspace{1cm} (14.a)

where \( t_{d}(\rho, \phi) \) is the time delay due to the spatial distance that the incident wave is traveling through, and \( t_{o} \) is the time when the pulse maximum occurs. When the Gaussian
pulse is applied to model line source problems, $t_d$ (the spatial time delay) should be set to be zero. Also, $T$ can be defined as [9]

$$T = \frac{10}{\sqrt{3}} \frac{\Delta d}{v_p}$$  \hspace{1cm} (14.b)

In equation (14b), $\Delta d$ is the smallest distance the wave passes through during one time step, and $v_p$ is the maximum velocity of propagation in a cell [7]. According to [9], the maximum usable frequency for the Fourier transformed data can be defined as

$$f_{\text{max}} = \frac{1}{2T}$$  \hspace{1cm} (14.c)

One of the advantages of using a Gaussian pulse as an excitation is that the Fourier transform of the Gaussian pulse retains in the same shape and is symmetric about the origin of the frequency spectrum. Another advantage is that by using (14c) the maximum frequencies in related frequency spectrums are well defined.

Furthermore, a sine wave function can also be an effective source model in the cylindrical FDTD, which can be defined as

$$\text{Sine}(\rho, \phi, n\Delta t) = \sin \left[ 2 \pi f (n\Delta t - t_d) \right]$$  \hspace{1cm} (15)

where $f$ is the propagating frequency and $t_d$ is the time delay.
Since the sine wave function is a continuous wave source, the field distributions will reach a steady state after a number of time steps. Therefore, comparison of examples to analytical solutions or other frequency domain numerical solutions can be performed.

2.4.2 Determination of Amplitudes and Time Delays for The Plane Wave Source

When Gaussian pulse and sine wave functions are used to model plane wave sources, the amplitudes of the incident wave components and the related spatial time delays need to be defined. The general incident field equations in spherical coordinates can be defined as [7]

\[ \vec{E} = \left[ E_\theta \hat{\theta} + E_\phi \hat{\phi} \right] f( n \Delta t - t_o - t_d ) \]  
\[ \vec{H} = \left[ \frac{E_\phi}{\eta} \hat{\theta} - \frac{E_\theta}{\eta} \hat{\phi} \right] f( n \Delta t - t_o - t_d ) \]  

where \( \hat{\theta} \) and \( \hat{\phi} \) are the unit vectors in the spherical coordinates, and \( f( n \Delta t - t_o - t_d ) \) can be either a Gaussian pulse or a sine wave function. For a sine wave function, \( t_o \) can be always set to be zero. Using the spherical to rectangular coordinate transformation, the amplitudes of each electric and magnetic incident wave can be expressed as

\[ E'_{\infty} = E_\theta \cos \theta \cos \phi - E_\phi \sin \phi \]  

(17.a)
\[ E'_{y_0} = E_\theta \cos \theta_1 \sin \phi_1 + E_\phi \cos \phi_1 \]  
(17.b)

\[ E'_{z_0} = -E_\theta \sin \theta_1 \]  
(17.c)

\[ H'_{x_0} = \frac{1}{\eta} \left( E_\theta \sin \phi_1 + E_\phi \cos \theta_1 \cos \phi_1 \right) \]  
(17.d)

\[ H'_{y_0} = \frac{1}{\eta} \left( -E_\theta \cos \phi_1 + E_\phi \cos \theta_1 \sin \phi_1 \right) \]  
(17.e)

\[ H'_{z_0} = \frac{1}{\eta} \left( -E_\phi \sin \theta_1 \right) \]  
(17.f)

where \( \eta \) is the wave impedance of the medium which can be expressed as

\[ \eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{1}{\varepsilon v} = \mu v \]  
(17.g)

Applying a rectangular to cylindrical coordinate transformation with \( \theta_1 = 90^\circ \), \( E_\theta = H_0 \eta \), and \( E_\phi = 0 \) to the incident wave amplitude in equation (17s), the incident wave equations for TE\( ^x \) polarization are given as

\[ H'_z(\rho, \phi, n\Delta t) = H_o \ f( n\Delta t - t_o - t_d ) \]  
(18.a)

\[ E'_\rho(\rho, \phi, n\Delta t) = \frac{-H_o}{\varepsilon v} \left( \cos \phi \sin \phi_1 - \sin \phi \cos \phi_1 \right) f( n\Delta t - t_o - t_d ) \]  
(18.b)

\[ E'_\phi(\rho, \phi, n\Delta t) = \frac{H_o}{\varepsilon v} \left( \cos \phi \cos \phi_1 + \sin \phi \sin \phi_1 \right) f( n\Delta t - t_o - t_d ) \]  
(18.c)

where \( H_o \) is the amplitude of the magnetic incident wave, \( v \) is the wave speed, and \( t_d \) is the spatial time delay which is a function of \( \rho \) and \( \phi \).
Similarly, using a rectangular to cylindrical coordinate transformation, with \( \theta_i = 90^\circ \), \( E_\theta = 0 \), and \( E_0 = E_o \eta_i \), the incident wave equations for TM\(^*\) polarization can be shown to be

\[ E_z^i(\rho, \phi, n\Delta t) = E_o f(n\Delta t - t_o - t_d) \]  
\[ (18.d) \]

\[ H_\rho^i(\rho, \phi, n\Delta t) = \frac{E_o}{\mu v} \left( \cos \phi \sin \phi_i - \sin \phi \cos \phi_i \right) f(n\Delta t - t_o - t_d) \]  
\[ (18.e) \]

\[ H_\phi^i(\rho, \phi, n\Delta t) = \frac{-E_o}{\mu v} \left( \cos \phi \cos \phi_i + \sin \phi \sin \phi_i \right) f(n\Delta t - t_o - t_d) \]  
\[ (18.f) \]

The spatial time delay, \( t_d \), is another important factor for an incident plane wave source, which is due to the distance \( d \) between the referenced origin \( O' \) and the projection of the testing point TP along the wave vector (as shown in Fig. 2). Therefore, the spatial time delay \( t_d \) can be expressed as [9]

\[ t_d = \frac{d}{v} = \frac{\hat{k}_{inc} \cdot \vec{r}'}{v} \]  
\[ (19.a) \]

where \( \hat{k}_{inc} \) is the unit incident wave vector, \( \vec{r}' \) the distance between the testing point TP and the referenced origin \( O' \) of the incident plane wave. \( \hat{k}_{inc} \) and \( \vec{r}' \) can be defined in cylindrical coordinate as follows

\[ \hat{k} = \hat{x} \cos \phi_i + \hat{y} \sin \phi_i \]  
\[ (19.b) \]
Fig. 2 Spatial distances and coordinate origins in cylindrical coordinates used for the determination of the time delay as associated with the incident plane wave equations.
\[
\vec{r}' = \vec{r} - \vec{O}' = \rho (\hat{x} \cos \phi + \hat{y} \sin \phi) + R (\hat{x} \cos \phi + \hat{y} \sin \phi)
\]  
(19.c)

where R is the extent of the computational domain in \( \rho \) direction. By integrating (19.a-c), the time delay \( t_d \) can be expressed as

\[
t_d = \frac{\rho (\cos \phi \cos \phi + \sin \phi \sin \phi) + R}{v}
\]  
(19.d)

Using equations (18) and (19), the incident plane wave excitations in two-dimensional cylindrical coordinates can be developed.

### 2.5 Artificial Absorbing Boundaries for Waveguides and Scattering Problems

An outer radiation boundary is used to solve the truncation problems for finite computational domains. At the boundary of the computational domain, wave behavior in this layer should not be affected by the truncation of the computational domain.

For the studies of cutoff frequencies inside circular or coaxial waveguides, the outer boundary is simplified by using a circular PEC wall around the computational domain. On that boundary, the tangential electrical field components are set to be zero.

For scattering problems, updated fields in the layers before the outer radiation boundary layers should be modeled as waves that are propagating continuously without any reflection. In this study, three categories of artificial ABCs are investigated: Mur finite difference absorbing scheme [1], Yee's tapered damping function [10], and matched layers technique [12,14]. Traditionally, a Mur outer radiation boundary is used in most
applications. For simplicity, the outer radiation boundary can be modeled by adding 10 to
20 layers of lossy materials [10], which is referred to as Yee's absorbing boundary
(Yee's ABC). The other technique, matched layers (ML) [12], is similar to Yee’s ABC.

The Mur finite difference scheme is one of the most effective absorbing boundary
conditions, and starts from the finite difference approximation of the scalar wave equation
as
\[
\frac{\partial^2 U}{\partial t^2} = c^2 \nabla^2 U \quad (20.a)
\]
where U is \(H_e\) for the TE\(^2\) polarization, and \(E_z\) for the TM\(^2\) polarization. In cylindrical
coordinates, equation (20.a) can be expressed as [13]
\[
\left( \frac{\partial}{c \partial t} + \frac{1}{\rho} \right) \frac{\partial}{\partial \rho} U = -\frac{\partial^2}{\partial \rho^2} U - \frac{3}{2\rho} \frac{\partial}{\partial \rho} U - \frac{3}{8\rho^2} U + \frac{1}{2\rho^2} \frac{\partial^2}{\partial \phi^2} U \quad (20.b)
\]
Using a finite difference approximation to equation (20.b), we have [13]
\[
\left( 1 + \frac{\Delta \rho}{c \Delta t} + \frac{3\Delta \rho}{4\rho_i} \right) U^n(N_{\rho},J) = \left( 1 - \frac{\Delta \rho}{c \Delta t} - \frac{3\Delta \rho}{4\rho_i} \right) U^{n-1}(N_{\rho}-1,J) + \left( 1 - \frac{\Delta \rho}{c \Delta t} + \frac{3\Delta \rho}{4\rho_i} \right) U^{n-2}(N_{\rho},J)
+ \left( -1 - \frac{\Delta \rho}{c \Delta t} + \frac{3\Delta \rho}{4\rho_i} \right) U^{n-2}(N_{\rho}-1,J) - \frac{2c\Delta t}{\rho_i} \left( U^{n-1}(N_{\rho},J) - U^{n-1}(N_{\rho}-1,J) \right)
+ \left( 2\frac{\Delta \rho}{c \Delta t} - \frac{3c\Delta t \Delta \rho}{8\rho_i^2} - \frac{c\Delta t \Delta \rho}{\rho_i^2 \Delta \phi^2} \right) \left( U^{n-1}(N_{\rho},J) + U^{n-1}(N_{\rho}-1,J) \right)
+ \frac{c\Delta t \Delta \rho}{2\rho_i^2 \Delta \phi^2} \left( U^{n-1}(N_{\rho},J+1) + U^{n-1}(N_{\rho},J-1) + U^{n-1}(N_{\rho}-1,J+1) + U^{n-1}(N_{\rho}-1,J-1) \right) \quad (20.c)
\]
where \(\rho_i\) and the spatial arrangement of U is shown in Fig. 3.
Fig. 3 The spatial arrangements of the parameters for Mur absorbing boundary condition in the cylindrical coordinate system.
Yee's absorbing boundary condition is built by extending 10 to 20 gradually increasing electric and magnetic lossy conformal layers to the computational domain [10]. According to the updating equations (4), (6), (9), and (12), at each absorbing layer there are absorbing factors with values between 0 and 1 for both electric and magnetic fields \((\varepsilon/(\varepsilon + \sigma \Delta t))\) for electric fields and \((\mu/(\mu + \sigma^* \Delta t))\) for magnetic fields. Therefore, Yee’s absorbing boundary is modeled such that at each absorbing layer, fields are updated first, and then the updated fields are multiplied by an absorbing factor, called a tapered damping function [10]. The tapered damping function can be defined as

\[
\text{Tap}(I) = \cos \left[ \frac{\pi}{3} \times \frac{(N + 1 - I)}{N} \right]
\]

where the layer number \(I = N, \ldots, 2, 1, 0\).

At the first layer of Yee’s ABC, the layer after the computational domain, \(I\) is equal to \(N\). If \(N\) is chosen between 10 and 20, \(\text{Tap}(1) = \cos\left( \frac{\pi}{3N} \right)\), which reduces reflections from the first ABC layer [10 and 12]. After \(N\) layers, the fields passing through the ABC layers are rather small. Also, it is recommended that the distance between scatterers and the Yee’s outer absorbing boundary be at least one wavelength [10]. When the number of absorbing layers is enough (at least 10 layers), the tapered damping function can be replaced by the square of tapering damping function.
The other absorbing boundary applied in this thesis is the matched layers as described in [14]. Instead of assigning a tapered damping function for the absorbing layers, the electric and magnetic conductivities of each absorbing layer are defined according to the wave impedance matched condition [14] as shown in equation (23.d). Furthermore, the values of conductivities in the absorbing layers can also be chosen subject to the size of a scatterer and the number of absorbing layers used [12]. According to [12], the reflections from the absorbing boundaries are caused primarily by the conductivities of the vacuum-layer interface (the interface between the computational domain and the first absorbing layer) and the incremental rate of the conductivities in the absorbing layers. Three types of mathematical functions can be used to model the conductivity variations: linear, parabolic, and geometrically increasing exponential functions [12]. In this thesis, a geometrical increasing exponential function was applied.

The spatial relations of the parameters used in the ML method are shown on Fig. 4. The derivation of the ML formulation begins by defining the reflection factor as [12]

\[ R(\theta) = \left[ R(0) \right]^{\cos \theta} \]  \hspace{1cm} (22.a)

where \( R(0) \) is the normal reflection factor defined as

\[ R(0) = e^{-\left(2/\epsilon_{\text{e}}\epsilon_0\right) \int_{\theta}^{\theta} \sigma(\rho) d\rho} \]  \hspace{1cm} (22.b)

Geometrically growing conductivities can be defined as

\[ \sigma(\rho) = \sigma_0 \left( \frac{g^{1/\Delta \rho}}{g^{1/\Delta \rho}} \right)^{\rho} \]  \hspace{1cm} (22.c)

where \( \sigma_0 \) is the conductivity in the vacuum-layer interface, and \( g \) is a factor assigned for geometrically increasing conductivities.
Fig. 4 Spatial arrangements of parameters used in cylindrical matched layers absorbing boundary condition.
Equation (22.c) can be expressed at the layer index $L$ as

$$
\sigma_n(L) = \frac{1}{\Delta \rho} \int_{\rho(L)-\Delta \rho/2}^{\rho(L)+\Delta \rho/2} \sigma(u) \partial u
$$

(22.d)

For an N-cell layer, the normal reflection factor $R(0)$ can be expressed as

$$
R(0) = e^{-2L/\varepsilon_o c} \left( \frac{gN - 1}{\ln g} \right) \varepsilon_o \Delta \rho
$$

(22.e)

By taking the natural logarithm of equation (22.e), the vacuum-layer conductivity can be expressed as

$$
\sigma_o = -\frac{\varepsilon_o c}{2 \Delta \rho} \ln g \frac{\ln R(0)}{g^N - 1}
$$

(23.a)

where the values of $R(0)$ are often used between 1 and 0.01 percent [12], which implies a very small reflection from the absorbing boundary. By integrating equation (22.d) and (22.e), the electrical conductivities of each layer are given as

$$
\sigma_n(0) = \sigma_o \frac{\sqrt{g} - 1}{\ln g} = \frac{\varepsilon_o c (1 - \sqrt{g}) \ln R(0)}{2 \Delta \rho (g^N - 1)}
$$

(23.b)

$$
\sigma_n(L > 0) = \sigma_o \frac{g - 1}{\sqrt{g} \ln g} \frac{g^L}{g^L} = \frac{\varepsilon_o c (1 - g) \ln R(0)}{2 \Delta \rho \sqrt{g (g^N - 1)}} \frac{g^L}{g^L}
$$

(23.c)

The magnetic conductivities of each layer can be obtained by using the matched impedance condition

$$
\frac{\sigma_o}{\varepsilon_o} = \frac{\sigma_o^*}{\mu_o}
$$

(23.d)
To choose the geometrical factor \( g \), an equation obtained by statistical curve regression of numerical experiments [12] was used as a reference, which is

\[
\log(g) = \alpha + \beta \log(N_a) + \gamma \left[ \log(N_a) \right]^2
\]  

(24.a)

where \( N_a \) is the number of cells of a PEC plate, a PEC square, or a PEC cross. When the numerical error is 1\%, the factors \( \alpha, \beta, \) and \( \gamma \) are 1.5851, -1.0156, and 0.18495, respectively. If numerical error is 3 \%, the factors \( \alpha, \beta, \) and \( \gamma \) are 1.7842, -1.0433, and 0.17749, respectively[12]. Although the geometries of scatterers used in this thesis are circular PEC cylinders, equation (24.a) is used as an approximation.

Using the geometrical factor \( g \) obtained by equation (24.a) and rewriting (23.b), the optimal number of absorbing layer is given by

\[
N = \frac{1}{\ln g} \ln \left[ 1 - \frac{\varepsilon_{sc}}{2} \frac{(1-\sqrt{g}) \ln R(0)}{\Delta \rho} \frac{1}{\sigma_x(0)} \right]
\]  

(24.b)

Applying equations (23) and (24) to determine the absorbing factors in the updating equations (4), (6), (9), and (12), a cylindrical ML absorbing layer can be built.

2.6 Summary

Using the updating equations described in section 2.2, the time step size described in section 2.3, the excitations presented in section 2.4, and one of outer absorbing boundaries described in section 2.5, programs for the z-invariant 2 dimensional CC-FDTD are designed and used for generating the numerical results as described in Chapter III.
CHAPTER III
NUMERICAL RESULTS

3.1 Introduction

To verify the 2-D cylindrical FDTD algorithms described in Chapter II, FORTRAN codes are built to compute and verify the numerical results. Total field updating equations (described in section 2.2.1) are applied to the study of cutoff frequencies inside coaxial waveguides and scattering from a PEC circular cylinder excited by near-zone line sources. Separate field updating equations (illustrated in section 2.2.2) are applied to compute the scattered fields from PEC circular cylinder illuminated by TE^{z} and TM^{z} normally incident plane waves. Three categories of artificial absorbing boundary conditions (presented in section 2.4) are also tested and compared in this chapter.

To illustrate the effectiveness of the numerical solutions from the two dimensional cylindrical FDTD technique, analytical solutions, and computational results obtained using traditional rectangular staircasing FDTD are presented and compared in several numerical examples. Root mean square errors of numerical results are presented from these numerical example so that the improvements of numerical results obtained using cylindrical FDTD can be seen.

Computational resource efficiency is also an important evaluation criterion of a numerical method. In this thesis, computation spaces (cell numbers) used in rectangular staircasing FDTD and cylindrical FDTD based on certain accuracy of computational results are also compared and evaluated to see whether or not there are any improvements by implementing cylindrical cells.
3.2 Cutoff Frequencies of Coaxial Waveguides

In this section, examples for the cutoff frequencies inside coaxial waveguides for TE\textsuperscript{z} and TM\textsuperscript{z} modes are presented based on computations for validating the formulation described in Chapter II. Traditionally, coaxial waveguides are excited by a voltage source between the center conductor and the outer metal shell or wall. If a coaxial waveguide is excited by a line current source in the z-direction inside the waveguide (electrical current source for TM\textsuperscript{z} mode and magnetic current source for TE\textsuperscript{z} mode), then the field components inside the coaxial waveguide are invariant with respect to the z-direction. Thus, the computations of fields inside coaxial waveguides to evaluate the cutoff frequencies will be greatly simplified and a two dimensional scattering CC-FDTD method can be used. For computing TM\textsuperscript{z} or TE\textsuperscript{z} mode cutoff frequencies of a coaxial waveguide (shown in Fig. 5 and Fig. 11, respectively), an assigned initial value of field component (E\textsubscript{z} for TM\textsuperscript{z} case and H\textsubscript{z} for TE\textsuperscript{z} case) in one time step (like a rectangular narrow pulse) which contains a very wide frequency bandwidth is used for the current source inside a waveguide (as shown in Fig. 3(a) for TM\textsuperscript{z} mode and in Fig. 5(a) for TE\textsuperscript{z} mode, where both are shown with 10° cell angle).

To obtain frequency domain results, the time domain responses at testing points are transformed to the frequency domain by using the fast Fourier transform (FFT) technique. In frequency domain, the magnitudes of the field components at cutoff frequencies are much stronger than those at other frequencies (as shown in Figs. 7-10 and Figs. 13-16). In this transformation algorithm, the number of time steps must be large...
enough [4] to ensure that the FFT results are appropriate, and, for this analysis, 32768 time steps are used for TM$^z$ and TE$^z$ cases in this study.

By tabulating all the peaks formed by the FFT in the frequency domain, the cutoff frequencies are attained and can be compared to the analytical solutions computed using the waveguide and cavity software WGC [5]. Because individual field components at some testing points may be very weak at certain cutoff frequencies when determining a mode, all the field components and fields at different testing points should be collected for comparison. The testing points chosen for TM$^z$ mode cutoff frequencies inside the coaxial waveguide are at (7.5 cm,300°), (7.5 cm,270°), (7.5 cm,180°), and those for TE$^z$ mode cutoff frequencies are at (1.25 cm,300°), (1.25 cm,270°), (1.25 cm,180°). Although the sizes of the waveguide for TM$^z$ and that for TE$^z$ are different, the ratio of inner and outer radii for both waveguides are the same. Information describing the inhomogeneous properties of cell sizes is presented in Table 2.

For computing TM$^z$ mode cutoff frequency, the field components inside a coaxial waveguide are reduced to $E_z$, $H_\phi$, and $H_\theta$ as shown on Fig. 5. The coaxial waveguide dimensions are a 5 cm inner radius and a 20 cm outer radius. The time domain response is shown in Fig. 6(b), and the magnitudes of field components in the frequency domain are shown in Fig. 7-9, where $\Delta\phi = 10^\circ$, 5°, and 1° are used, respectively. The cutoff frequencies are also shown in Table 1(a) along with the analytical values from [5] and values obtained using the rectangular staircasing FDTD technique.
Fig. 5 Field components for TM$^e$ mode in a transverse plane cut inside a coaxial waveguide.
Fig. 6 Time domain responses of (a) a z-directed current $J_z$ (b) $E_z$ inside a coaxial waveguide with inner radius 5 cm and outer radius 20 cm. (Measured at the source point.)
Fig. 7-1 TM$^\prime$ mode cutoff frequencies inside a coaxial waveguide computed by CC-FDTD with 10° cell angle (a) $E_z$ component (b) Combined magnetic fields.
Fig. 7-2 TM$^\prime$ mode cutoff frequencies inside a coaxial waveguide computed by CC-FDTD with $10^\circ$ cell angle (a) $H_\rho$ component (b) $H_\theta$ component.
Fig. 8-1 TM' mode cutoff frequencies inside a coaxial waveguide computed by CC-FDTD with 5° cell angle (a) $E_z$ component (b) Combined magnetic fields.
Fig. 8-2  TM\textsuperscript{r} mode cutoff frequencies inside a coaxial waveguide computed by CC-FDTD with 5° cell angle (a) $H_p$ component (b) $H_q$ component.
Fig. 9-1  TM^2 mode cutoff frequencies inside a coaxial waveguide computed by CC-FDTD with 1° cell angle (a) $E_z$ component (b) Combined magnetic fields.
Fig. 9-2  TM$^2$ mode cutoff frequencies inside a coaxial waveguide computed by CC-FDTD with 1° cell angle (a) $H_p$ component (b) $H_\theta$ component.
Fig. 10-1 TM² mode cutoff frequencies inside a coaxial waveguide computed by staircasing FDTD (a) $E_z$ component (b) Combined magnetic fields.
Fig. 10-2  TM$^2$ mode cutoff frequencies inside a coaxial waveguide computed by staircasing FDTD  (a) $H_x$ component (b) $H_y$ component.
Table 1(a) The TM_{mn}^z mode cutoff frequencies (GHz) inside a coaxial waveguide with inner radius 5 cm and outer radius 20 cm.

<table>
<thead>
<tr>
<th>Mode m n</th>
<th>Analytical (WGC)</th>
<th>Staircasing FDTD 32,768 steps</th>
<th>CCFDTD $\Delta\phi = 10^\circ$ 32,768 steps</th>
<th>CCFDTD $\Delta\phi = 5^\circ$ 32,768 steps</th>
<th>CCFDTD $\Delta\phi = 1^\circ$ 32,768 steps</th>
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<td>0.980</td>
<td>0.980</td>
<td>0.989</td>
</tr>
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<td>1.277</td>
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<td>2.037</td>
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</tr>
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<td>2.396</td>
<td>2.399</td>
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<td>2.647</td>
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</tr>
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<td>2.988</td>
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<td>2.991</td>
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<td>3.025</td>
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<td>3.024</td>
<td>3.022</td>
<td>3.022</td>
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<tr>
<td>23</td>
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<td>3.130</td>
<td>3.128</td>
<td>3.129</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>-</td>
<td>3.174</td>
<td>3.228</td>
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<td>33</td>
<td>3.300</td>
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<td>3.296</td>
<td>3.297</td>
<td>3.299</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>-</td>
<td>3.423</td>
<td>-</td>
<td>-</td>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>3.657</td>
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<td>-</td>
<td>-</td>
<td>3.900</td>
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<td>3.831</td>
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<td>04</td>
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<td>1440</td>
<td>2880</td>
<td>14400</td>
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<tr>
<td>Root Mean Square Errors (%)</td>
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<td>1.521</td>
<td>0.648</td>
<td>0.179</td>
<td>0.693</td>
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</table>
For computing TE\(^2\) mode cutoff frequencies, the procedures are similar to those used for TM\(^2\) mode where the field components inside a coaxial waveguide can be reduced to \(H_z\), \(E_\rho\), and \(E_\phi\) as shown in Fig. 11. The coaxial waveguide dimensions are 0.5 cm inner radius and 2 cm outer radius. The time domain response is shown in Fig. 12(b), and the magnitudes of the field components in the frequency domain are shown in Fig. 5(c)-(e). The cutoff frequencies are also presented in Table 1(b).

In Table 1(a) and 1(b), analytical values of cutoff frequencies within certain frequency ranges are used as a criterion to evaluate the accuracy of each FDTD method. Root square mean errors of all cutoff frequencies are performed to evaluate each FDTD method. The root square mean error (RMS error) is defined as

\[
\text{RMS error} = \sqrt{\frac{1}{N_c} \sum_{i=1}^{N_c} \left(\frac{\text{Numerical Solution} - \text{Analytical Solution}}{\text{Analytical Solution}}\right)^2}
\]  

(25.a)

where \(N_c\) is the total number of cutoff frequencies compared.

Because of the inherent inhomogeneous cell size used in CC-FDTD methods and non-smooth modeling of the boundaries of coaxial waveguide in the staircasing FDTD method, numerical errors are expected. The deviations of the numerical results in TE\(^2\) and TM\(^2\) case from analytical solutions are both shown in Fig. 17 and Tables 1(a)-(b).
For TM case, the cutoff frequencies are not very high, due to the geometry dimensions. Therefore, numerical results from the CC-FDTD and the staircasing FDTD are effective enough in comparison with analytical solutions. The deviations of the numerical results by CC-FDTD with various cell angles are smaller than those obtained from the staircasing FDTD approach.

For TE case, the geometry is one tenth of that in TM case. Therefore the cutoff frequencies in TE are higher than those in TM case. Deviations of the computed results from the staircasing FDTD are rather large as compared to those from CC-FDTD. In additions, some TE mode cutoff frequencies are not detected by staircasing FDTD.

According to Fig. 17 and Table 1, not only better accuracy, but also computer resource efficiency, is obtained using the CC-FDTD method as compared to staircasing FDTD analysis.
Fig. 11 Field components for TE² mode in a transverse plane cut inside a coaxial waveguide.
Fig. 12 Time domain responses of (a) a $z$-directed current $M_z$ (b) $H_z$ inside a coaxial waveguide with inner radius 0.5 cm and outer radius 2.0 cm.

(Measured at the source point.)
Fig. 13-1 TE\textsuperscript{z} mode cutoff frequencies inside a coaxial waveguide computed by CC-FDTD with 10° cell angle (a) Combined electric fields. (b) $H_z$ component.
Fig. 13-2 TE₂ mode cutoff frequencies inside a coaxial waveguide computed by CC-FDTD with 10° cell angle (a) $E_\rho$ component (b) $E_\phi$ component.
Fig. 14-1  TE² mode cutoff frequencies inside a coaxial waveguide computed by CC-FDTD with 5° cell angle  (a) Combined electric fields.  (b) Hₓ component.
Fig. 14-2 TE$^2$ mode cutoff frequencies inside a coaxial waveguide computed by CC-FDTD with 5° cell angle (a) $E_\phi$ component (b) $E_\theta$ component.
Fig. 15-1  TE² mode cutoff frequencies inside a coaxial waveguide computed by CC-FDTD with 1° cell angle (a) Combined electric fields. (b) \( H_z \) component.
Fig. 15-2  TE\textsuperscript{2} mode cutoff frequencies inside a coaxial waveguide computed by CC-FDTD with 1° cell angle (a) $E_\rho$ component (b) $E_\phi$ component.
Fig. 16-1  TE\(^2\) mode cutoff frequencies inside a coaxial waveguide computed by staircasing FDTD (a) Combined electric fields. (b) H\(_z\) component.
Fig. 16-2 TE² mode cutoff frequencies inside a coaxial waveguide computed by staircasing FDTD (a) $E_x$ component (b) $E_y$ component.
Fig. 17 Comparisons of analytical cutoff frequencies and numerical results by various FDTD methods (a) TM² case (b) TE² case.
Table 1(b)  The $TE_{mn}^z$ mode cutoff frequencies (GHz) inside a coaxial waveguide with inner radius 0.5 cm and outer radius 2.0 cm.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Analytical Results</th>
<th>Staircasing FDTD 32,768 steps</th>
<th>CCFDTD $\Delta\phi = 10^\circ$ 32,768 steps</th>
<th>CCFDTD $\Delta\phi = 5^\circ$ 32,768 steps</th>
<th>CCFDTD $\Delta\phi = 1^\circ$ 32,768 steps</th>
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<tr>
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<td>3.923</td>
<td>-</td>
<td>3.838</td>
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<td>3.938</td>
</tr>
<tr>
<td>21</td>
<td>7.179</td>
<td>-</td>
<td>7.107</td>
<td>7.124</td>
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<td>31</td>
<td>10.00</td>
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<td>10.523</td>
<td>10.520</td>
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<td>17.727</td>
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<td>-</td>
<td>19.460</td>
<td>-</td>
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<td>02</td>
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<td>20.003</td>
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<th>40 x 72</th>
<th>40 x 360</th>
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</thead>
<tbody>
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<td>Total Cells</td>
<td>-</td>
<td>10000</td>
<td>1440</td>
<td>2880</td>
<td>14400</td>
</tr>
</tbody>
</table>

| Root Square Mean Errors (%) | - | 4.396 | 2.055 | 1.613 | 1.454 |

Table 2  The cell sides length ratios ($L/\Delta \rho$) of innermost and outermost cells inside with various values of cell angle ($\Delta \phi$).

<table>
<thead>
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<th>Cell Angle $\Delta \phi$ Used</th>
<th>10°</th>
<th>5°</th>
<th>1°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Innermost Cell</td>
<td>1.7453</td>
<td>0.8727</td>
<td>0.17453</td>
</tr>
<tr>
<td>Side Length Ratio</td>
<td>1.7453</td>
<td>0.8727</td>
<td>0.17453</td>
</tr>
<tr>
<td>Outermost Cell</td>
<td>6.9813</td>
<td>3.4907</td>
<td>0.69813</td>
</tr>
<tr>
<td>Side Length Ratio</td>
<td>6.9813</td>
<td>3.4907</td>
<td>0.69813</td>
</tr>
</tbody>
</table>
3.3 Segmentation and Amplitudes of Responses Excited by a Near Zone Source

Before discussing the scattering problems from circular scatterers that are excited by a near zone source, a numerical example is given to illustrate the relation between cell segmentations and amplitudes of the responses.

In Fig. 18, there are three kinds of cell segmentations. The cell side of segmentation 3 is twice of that of segmentation 1, and the cell side of segmentation 2 is half of that of segmentation 1. In other words, the cell area of segmentation 3 is 4 times that of segmentation 1, and the cell area of segmentation 2 is a quarter of that of segmentation 1. With respect to total energy passing through the cell at the excitation point, the energy in the various cells are different, even though the same amplitudes and widths of excitations are adopted. When the amplitude and the width of the excitation are chosen, the energy through the cells at the excitation point with different segmentations is also determined by the area of the cells. In other words, the energy associated with a cell in segmentation 3 is 4 times of that in segmentation 1. Similarly, the energy associated with a cell in segmentation 2 is one quarter of that in segmentation 1.

When a scatterer excited by a near zone source with a certain width is modeled by different segmentations, matched numerical results can be obtained by simply adjusting the amplitudes of excitations. This makes the energy used for excitation in various segmentation to be approximately the same. This relation between the segmentations and the amplitudes of responses can be proven by the numerical example described in Fig. 18(b). Matched numerical results are shown in Fig. 18 that are obtained by adjusting the amplitudes of excitations.
Fig. 18 A numerical experiment used to prove the relation between cell segmentations and the amplitude of response excited by near zone sources. (a) Various cell segmentations at excitation points. (b) The geometry of the problem.
Fig. 19 A rectangular cylinder excited by a z-directed electric current.
3.4 Scattering from Circular Cylinder Excited by a Near Zone Source

With the knowledge described in section 3.2, the comparison of the scattering from a circular cylinder excited by a near zone source can be made correctly. To validate the scattering formulation and ABC of CC-FDTD, a z-invariant PEC scatterer with radius of 1cm excited by a z-directed Gaussian pulse current, is used as described in Fig. 20. Four testing points are chosen for comparing the wave behavior computed by CC-FDTD and the staircasing FDTD.

The wave behavior at testing point 4 is so weak that it is difficult to be modeled correctly, because of the shadowing effect of the PEC scatterer. In Fig. 21, a TM scattering example is given. The numerical results by CC-FDTD and by staircasing FDTD methods are presented for comparisons that should provide insights of scattering in time domain. To obtain the same accuracy in this example, 142,884 cells are used in staircasing FDTD, but the number of cells used in CC-FDTD is 57,600, 40 percent of those used in staircasing FDTD. In other words, the computer resource efficiency of CC-FDTD is better than that of staircasing FDTD.

To prove the computational space efficiency of CC-FDTD is better than the staircasing FDTD, the electric field at testing point 2 are computed by the staircasing FDTD using different number of cells. As shown in Fig. 22, the computed fields obtained from staircasing FDTD methods are closer to the results obtained from CC-FDTD as the number of cells is increased.
Fig. 20 A PEC cylinder excited by a z-directed line source modeled using the staircasing FDTD and CC-FDTD.
Fig. 21 A circular PEC cylinder excited by a z-directed current source modeled by staircasing FDTD and CC-FDTD.
Fig. 22 Convergent test of $E_z$ at testing point 2 obtained from staircasing FDTD with different segmentations and CC-FDTD.
3.5 Scattering from a Circular Cylinder Excited by a Far Zone Source

Scattered fields from a PEC circular cylinder illuminated by a TE\textsuperscript{z} or a TM\textsuperscript{z} normally incident plane waves are computed to further verify the formulation in section 2.2.1 and 2.4. In this section, a sine wave function is used to model the incident plane wave. The numerical results are depicted in Fig. 23. Bistatic scattering fields from a PEC z-invariant cylinder with 1 m radius are measured at the radius of 2.48 m, which was used in [13]. After a certain number of time steps, the field distribution will reach steady state. Fields at cells on the circle with 2.48m radius are determined. The results of this numerical example can be compared with those obtained using other numerical methods, such as Method of Moments (MoM). Analytical solutions are also used to verify the accuracy of CC-FDTD data. The scattered fields of TE\textsuperscript{z} and TM\textsuperscript{z} excitation are shown in Fig. 24 and Fig. 25, respectively. The root mean square error of all testing points and the dimensions required for each numerical example are shown in Table 3. This error quantity is defined as

\[
\text{RMS Error} = \sqrt{\frac{1}{N_t} \sum_{i=1}^{N_t} \left( \frac{\text{Numerical Solution} - \text{Analytical Solution}}{\text{Analytical Solution}} \right)^2}
\]  

(25.b)

where \(N_t\) is the total number of testing points.

The errors in these four examples are still acceptable, and the computational resources used are also rational. From these four numerical examples, computational
results of TM\(^z\) cases have better accuracy and need less computational space. In two dimensional TM\(^z\) cases, the spatial approximations of fields in \(\rho\) and \(\phi\) directions are magnetic fields instead of electrical fields in TE\(^z\) cases. Because the magnitudes of magnetic fields are much smaller than electric fields, better spatial approximations of magnetic fields can be made. Therefore, better accuracy and computer resource efficiency using a cylindrical FDTD are obtained in TM\(^z\) cases.
Fig. 23 A z-invariant PEC cylinder illuminated by TE$^z$ or TM$^z$ normal incident plane wave.
Fig. 24 Scattered magnetic fields from a 1 m radius PEC scatterer illuminated by a TE\(^2\) normal incident sine waves (a) Frequency = 150 MHz (b) Frequency = 300 MHz.
Fig. 25 Scattered magnetic fields from a 1 m radius PEC scatterer illuminated by a TM$_0$ normal incident sine waves (a) Frequency = 150 MHz (b) Frequency = 300 MHz.
Table 3(a) The numerical errors and number of cells used for the scattering examples with $\text{TE}^z$ plane wave illumination.

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</tr>
</thead>
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<tr>
<td><strong>Frequency</strong></td>
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<td><strong>Root Mean Square Error (%)</strong></td>
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<tr>
<td><strong>Total cells</strong></td>
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</table>

Table 3(b) The numerical errors and number of cells used for the scattering examples with $\text{TM}^z$ plane wave illumination.

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3.6 Analysis of The Absorbing Boundary Conditions

Numerical absorbing boundaries are one of the most important parts in the FDTD applications. Reflections from the numerical boundaries always affect the accuracy of FDTD method [1,12]. Therefore, finding an optimal outer absorbing boundary for the numerical system is also important. In this section, four numerical absorbing boundaries (Mur’s ABC, ML, Yee’s tapered damping function and squared Yee’s tapered damping function) are implemented to determine the effectiveness and efficiency of these absorbing boundaries in cylindrical FDTD.

A numerical example is computed using cylindrical FDTD with various absorbing boundaries. In this numerical experiment, a PEC cylinder with a radius of 1m is illuminated by a TM$^e$ 150 MHz plane wave. The computational space is modeled by 55 x 180 cells, and the cylinder is modeled by 25 x 180 cells. Fields are measured at 10 cells from the cylinder; in other words, 180 testing points are measured. A large number of time steps is used to see the effects of reflections from the absorbing boundaries. Analytical solutions are also used as a criterion, and global errors defined in equation (25.b) measured every 10 time steps after 200 time steps.

Mur’s absorbing boundary only needs one cell layer, so the distance between the scatterer and the absorbing layer is the maximum of these four absorbing boundaries. The more layers used in ML or Yee’s ABC, the smaller the distance between the scatterer and the absorbing boundary. To simplify the discussion, the absorbing boundaries will be described by the parameters used. For example, ML(15-10-G2.47-1) means a 15 cell absorbing layer, 10 cell distance between the testing point and absorbing layer, and a
lossy layer characterized by the geometrical factor 2.47 computed by the optimization equation (25.1) with 1% normal reflection. Yee’s ABC parameters can simply be described by the cell number of absorbing layer and the distance between the testing point and the first absorbing layer. The damping factors for electric and magnetic fields are shown in Fig. 26, and the absorbing factors at each layer of Yee’s tapered function and Yee’s tapered function square are presented in Fig. 26 and Fig. 27. The error analysis from 200 time steps (about 40 ns) to 1500 time steps (300 ns) of each absorbing boundary are shown in Fig. 28 to Fig. 30.

Since the space between the testing point and the absorbing layer is the maximum in the cylindrical Mur’s ABC, the errors from 200 to 300 time steps are larger than those after 300 time steps, because the fields in this computational domain have not reached the steady state. After 300 time steps, the RMS errors are from 0.16 % to 0.25 % (from -56 dB to -52 dB), which is the least error for cylindrical FDTD in TM$^\circ$ scattering problems of the ABCs studies here.

The space within the computational domain with ML and Yee’s ABC is less than that in cylindrical Mur’s case, so the fields become steady before 200 time steps, and the RMS errors in ML and Yee’s grow because of reflections from the boundaries.

The number of layers used with ML are 5 to 15. In Fig. 28 and Fig. 29, after 300 time steps, the errors for the ML using three different numbers of layer remain at the same level, around 3.5% to 5% (-28 to -26.5 dB), which indicate that as few as 5 absorbing layers can be used as an appropriate ML by choosing the conductivities according to equation (23) and (24). According to equation (24.1), the optimal number of ML layer is 14 layers. However, in this numerical experiment, the free space layers will
become fewer, which may cause reflections to happen earlier. However, in this experiment, ML absorbing boundaries using three different parameters have similar RMS errors, even if the free space region is reduced by large increments of cell numbers.

In this numerical experiment, reasonable results can be obtained by Yee’s tapered damping function and tapered damping function squared, only when the number of layers is greater or equal to 15 (shown in Fig. 29). Using 15 absorbing layers is the optimal case in both tapered damping function or tapered damping function square. The errors are increased as the number of layer-cells are reduced because of drastic decreases of damping factors (shown in Fig. 30). Reflections can be reduced by increasing the number of absorbing layers. However, in this experiment, reflections are also increased by the reductions of free space region between the scatterer and the ABC first layer in the computational domain. Therefore, the RMS errors from 15 layers are less than those of 20 layers.

Although the cylindrical Mur boundary is found to be the best one in these examples of cylindrical FDTD with TM cases, Mur’s scheme is not easily applied to arbitrary nonuniform grids. Therefore, matched layers can be an ideal alternative, which is easier to construct conformally to fit the scatterers’ geometries and for a certain accuracy. Additionally, the number of layers used in ML can be as small as 5 layers and can be located very close to the scatterer [12].
Fig. 26-1 Absorbing damping factors in ML. (a) For Electric fields (b) For Magnetic fields. ($g = 2.47, R(0) = 1\%$)
Fig. 26-2 Absorbing damping factors in ML. (a) For Electric fields (b) For Magnetic fields. ($g = 1.47, R(0) = 1\%$)
Fig. 26-3 Absorbing damping factors in ML. (a) For Electric fields (b) For Magnetic fields. (g = 1.47, R(0) = 0.01%)
Fig. 26-4 Absorbing damping factors in ML. (a) For Electric fields (b) For Magnetic fields. \( g = 3.47, R(0) = 1\% \)
Fig. 26-5 Absorbing damping factors in ML. (a) For Electric fields (b) For Magnetic fields. ($g = 3.47, R(0) = 0.01\%$)
Fig. 27 Absorbing damping factors for both electrical and magnetic fields. (a) Based on Yee's tapered damping function. (b) Based on the square of Yee's tapered damping function.
Fig. 28 RMS errors caused by absorbing boundary condition. (a) Cylindrical Mur's absorbing boundary condition. (b) ML absorbing boundary condition.
Fig. 29-1  RMS errors caused by absorbing boundary condition. (a) ML ($g = 1.47$, $R(0) = 1\%$). (b) ML($g = 1.47$, $R(0) = 0.01\%$).
Fig. 29-2  RMS errors caused by absorbing boundary condition. (a) ML (g = 3.47, \( R(0) = 1\% \)). (b) ML (g = 3.47, \( R(0) = 0.01\% \)).
Fig. 30 RMS errors caused by absorbing boundary condition. (a) Yee’s tapered damping function. (b) The square of Yee’s tapered damping function.
CHAPTER IV
CONCLUSIONS

For the bandwidth of cutoff frequencies within 4 GHz for the coaxial waveguide excited by a z-directed electric current (TM case), the best approximations are obtained for a 5° cell angle. The results for a 1° cell angle are the worst in CC-FDTD methods, but still better than staircasing FDTD. For the bandwidth of cutoff frequencies from 3 GHz to 20 GHz for the coaxial waveguide excited by a z-directed magnetic current (TE case), the best results are obtained for a 1° cell angle, and the results for a 10° are the worst. Therefore, for modeling waveguides that contain high cutoff frequencies, small angle differences should be chosen (1° or larger) to ensure acceptable accuracy. On the contrary, if the lower bandwidth (3 GHz- 4 GHz) is of concern, larger cell angle differences should be used (5° or larger).

As given in Figs. 17(a) and 17(b), the computational results of CC-FDTD method for various cell angle cases for this coaxial geometry are compared to each other. According to Table 1(a), the deviations in cutoff frequencies for the CC-FDTD methods are rather small as compared to the traditional staircasing FDTD method. Not only is the accuracy increased by CC-FDTD methods, but the computational domains are also significantly reduced for both 10° and 5° cell angle cases. For the TE case, since the size of waveguide is one tenth of the waveguide for the TM case, the cutoff frequencies are ten times larger than those of the waveguide used in the TM case. In addition, in the TE mode, some cutoff frequencies are very close to each other and very high (larger than 20 GHz). As shown in Fig. 6(b) and Table 1(b), the deviations in cutoff frequencies for the
CC-FDTD methods are still rather small (around 0.05), but the deviation of the results as compared to the staircasing FDTD is relatively large (around 0.1).

For $\text{TM}^\circ$ scattering from a circular cylinder excited by a near zone source, the accuracy of the computational results by CC-FDTD is highly acceptable which is proven from a comparison with the results from the staircasing FDTD. In addition, for CC-FDTD technique, only 40 percent of the cells that were used in staircasing FDTD are needed.

For the studies of scattered fields from a circular PEC cylinder excited by near-zone line source, comparable results can be obtained with cylindrical FDTD using 40 percent of the computational space of that required by staircasing FDTD. Very good accuracy and computational space efficiency are also provided by CC-FDTD as applied to computed scattered fields from a PEC cylinder illuminated by normal incident plane waves.

The implementations of various absorbing boundaries are also presented and evaluated by the use of numerical examples. Cylindrical Mur’s finite difference scheme was found to be the optimal one. Although the accuracy of matched layers (ML) technique is not as good as cylindrical Mur’s, ML is much easier to implement and stable even when the number of absorbing layers is reduced to 5 layers.

Therefore, based on this study, CC-FDTD methods not only increase accuracy but also provide time and computational resources efficiencies in comparison with respect to the Cartesian FDTD method, when the geometry fits the cylindrical coordinate system.
CHAPTER V

REFERENCES


VITAE

Chun-Wen Paul Huang was born in Taipei, Taiwan, R.O.C., on October 18, 1967. He received the diploma in Electronic Engineering from National Taipei Institute of Technology, Taipei, Taiwan, in 1991. From 1991 to 1993, he served in the army as a second lieutenant as the officer in charge of Telecommunications Techniques and Systems in Cashing Regiment Control Area Command, Cashing, Taiwan, R.O.C. After his service in the army, he served in a teaching position at Yung-Hsing Tutorial Center, Taipei, Taiwan, R.O.C. In August 1994, he began his work toward M.S. degree in engineering since-electrical engineering at The University of Mississippi. His research interests are in the analysis and design of microwave circuits, the electronics of telecommunications systems, and the applications of FDTD for solving electromagnetic problems.