Scattering by a double wedge and a parallel cylinder

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A simple technique for solving the problem of an E-polarized plane wave scattered by three objects consisting of two wedges and a cylinder is presented. The method is based on an extension of the Karp and Russek solution for the diffraction by a wide slit, and is applied here to investigate the transmission coefficient through the aperture of a wide double wedge in the presence of a circular dielectric or conducting cylinder whose axis is parallel to the edges of the two wedges. An improvement in the transmission coefficient is observed in the presence of a dielectric cylinder, whereas a conducting cylinder yields a lower transmission coefficient relative to the case when the cylinder is absent.

1. Introduction

A solution based on asymptotic approximation of the electromagnetic field scattered by these objects due to an incident E-polarized plane wave has not received great attention in the literature. A possible approach to use fictitious line sources, properly located according to the geometry of each scatterer, to account for the interaction fields between the scatterers. This technique was previously used by Clemenow (1956), Karp and Russek (1956) for the diffraction by a wide slit and recently extended by the authors (1984, 1985a) to solve for the diffraction by a wide double sharp or capped wedge. Hongo (1978) applied the technique to solve for the field scattered by two circular cylinders, whereas Ragheb and Hamid (1985) used the same concept for the scattering by an arbitrary number of cylinders. All previous results were restricted to two or more perfectly conducting bodies of the same type and dimensions (e.g. two half-planes, two wedges, two circular cylinders, N circular cylinders, etc.)

This paper shows that it is possible to apply the technique to two or more different scatterers provided that all of them are infinite along one of the coordinate axes, such as an infinite cylinder with arbitrary cross-section, half-plane, wedge with sharp, rounded, or capped edge, or any combination. In the present geometry, we consider a circular conducting or dielectric cylinder located half-way between the edges of a double sharp wedge or along the normal to the aperture plane of two sharp wedges at a distance d from the centre. Both the cylinder and the double wedge are assumed to be infinite in the z-direction. A similar loading of the full aperture of a slit by a circular dielectric cylinder was investigated by Hard and Sachdeva (1975) whose solution is restricted to narrow slit width (kr < 2.4!, where e is the relative permittivity of the cylinder), and yield a maximum error of 2.1% when e = 1.

In the present analysis, the interaction fields between the cylinder and the two wedges are clearly presented by simple relations using the known solutions for the scattered field by a wedge alone and a cylinder alone due to a plane wave incidence

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and a line source excitation. Furthermore, this type of solution offers much physical insight into the complex mechanism of multiple scattering between the three bodies.

It should be pointed out that any improvement in the diffraction or scattering behaviour of a double wedge (or a slit) using a third body could be useful, particularly for tandem or coupled apertures and aperture arrays. In such cases it is necessary to optimize with respect to shape, position and composition of the dielectric or metallic insert. This is particularly true in view of the advantages of using dielectric inserts to improve the behaviour of aperture antennas (Hamid and Al-Solaiman 1983) where the aperture may be formed from two wedges or two half-planes.

2. Formulation

The geometry of the problem is shown in Fig. 1, where a circular cylindrical coordinate system $(\rho, \phi, z)$ is used. The problem is two-dimensional since all fields are uniform in the $z$ direction. We consider the E-polarization case where the electric field has a $z$ component only and the $\exp(\mathrm{i}wt)$ time dependence is suppressed throughout.

A conducting wedge geometry may be defined by the half-planes at $\phi = \gamma$ and $\phi = 2\pi - \gamma$ intersecting along the $z$ axis. For a line source of unit amplitude at $(\rho_0, \phi_0)$ and parallel to the $z$ axis, the total field in the presence of the wedge is the incident field $E^i$ plus the scattered field $E^s$. $E^i$ is given by

$$ E^i = \frac{\pi}{2\rho} H_d(kR) $$

(1)

After some mathematical manipulations $E^s$ can be written in the following form

$$ E^s = \frac{\pi}{2\rho} H_d(k\rho) \exp[\mathrm{i}k\rho_0 \cos(\phi - \phi_0)] $$

(2)

where $R$ is the distance between the line source and the field point, $k$ is the wavenumber $2\pi/\lambda$, $\lambda$ is the wavelength and $H_d(x)$ is the Hankel function of the second

![Figure 1. Geometry of the problem.](image)
kind of order zero and argument \(x\). Using the exact series solution of the total field due to a line source near a conducting sharp wedge (James 1980), \(E_\text{w}^w\) is found to be

\[
E_\text{w}^w = \frac{\pi}{2j} \mathcal{H}_0(k\rho) f_\text{w}(\phi, \rho_0, \phi_0, \gamma)
\]

where the superscript \(w\) refers to the wedge and

\[
f_\text{w}(\phi, \rho_0, \phi_0, \gamma) = -\exp\left[ -j k \rho_0 \cos(\phi - \phi_0) \right] \sum_{n=1}^{\infty} \phi_n J_n(k\rho_0) \sin \left( \frac{n}{v} (\phi - \gamma) \right) \sin \left( \frac{n}{v} (\phi_0 - \gamma) \right)
\]

\[
e = 2(\pi - \gamma)/\pi
\]

while the asymptotic expression of the Hankel function is replaced by the Hankel function itself in the expression of \(E_\text{w}^w\).

For a plane wave of unit amplitude incident on the wedge at angle \(\phi_0\) with respect to the negative \(x\) axis, \(E^i\) is given by

\[
E^i = \exp(ik\rho \cos(\phi - \phi_0))
\]

Since the diffracted field (Keller 1975) is the same as the scattered field in the forward direction, \(E_\text{w}^w\) due to an incident plane wave can be written as

\[
E_\text{w}^w = \frac{\pi}{2j} \mathcal{H}_0(k\rho) g_\text{w}(\phi, \rho_0, \gamma)
\]

where

\[
g_\text{w}(\phi, \rho_0, \gamma) = \sum_{n=1}^{\infty} \left( \frac{\cos \left( \frac{n}{v} \phi \right) - \cos \left( \frac{n}{v} \phi_0 \right) }{v} \right)^{-1} \left( \frac{\cos \left( \frac{n}{v} \gamma \right) - \cos \left( \frac{n}{v} \phi + \phi_0 - \frac{2\pi}{v} \right) }{v} \right)^{-1}
\]

A circular cylinder is defined by the surface \(\rho = a\), while its axis coincides with the \(z\) axis. The scattered fields due to a line source and a plane wave incident on a circular cylinder (Harrington 1961) are given by \(E_\text{w}^c\) and \(E^c\), respectively, and are rewritten in the following modified form

\[
E_\text{w}^c = \frac{\pi}{2j} \mathcal{H}_0(k\rho) f(\phi, \rho_0, \phi_0, a)
\]

(6)

\[
E^c = \frac{\pi}{2j} \mathcal{H}_0(k\rho) g(\phi, \rho_0, a)
\]

(7)

while the superscript \(c\) refers to the cylinder, and \(f\) and \(g\) are the scattered field patterns due to a line source field and a plane wave incident, respectively. For a conducting cylinder, the \(f\) and \(g\) functions in (6) and (7) are denoted by \(f_\text{c}\) and \(g_\text{c}\), respectively, where

\[
f_c(\phi, \rho_0, \phi_0, a) = -\sum_{n=0}^{\infty} \epsilon_n \frac{j_n(k\rho_0)}{k_n(k\rho_0)} \cos(n\phi + \phi_0)
\]

\[
g_c(\phi, \phi_0, a) = -\frac{2j}{\pi} \sum_{n=0}^{\infty} \epsilon_n \frac{j_n(k\rho_0)}{k_n(k\rho_0)} \cos(n\phi + \phi_0)
\]

whereas, for a dielectric cylinder with relative permittivity \(\epsilon_r\), the corresponding
scattered field patterns $f_3$ and $g_3$ due to a line source field and a plane wave incident are given, respectively, by

$$
f_3(\phi, \rho_3, \phi_3, a, \epsilon) = -\sum_{\epsilon=0}^{\infty} \epsilon \varepsilon_p T_e H_2(k_0 \rho_3) \cos[n(\phi - \phi_3)]
$$

$$
g_3(\phi, \phi_3, a, \epsilon) = \frac{1}{2} \sum_{\epsilon=0}^{\infty} \varepsilon \epsilon - 1)^2 T_e \cos[n(\phi - \phi_3)]
$$

where

$$
T_e = \frac{k_1 J_{n-1}(k_0 r)}{k_0 J_n(k_0 r)} - k_1 J_{n-1}(k_0 r) J_n(k_0 r) / k_0 J_n(k_0 r) - k_1 H_{n-1}(k_0 r) / k_0 H_{n}(k_0 r)
$$

while $k_1 = \sqrt{(n)k_0}$ and the prime indicates differentiation with respect to the full argument. In the above equations the Neumann number $\varepsilon_e = 1$ for $n = 0$ and 2 for $n > 0$. $J_n(x)$ is the Bessel function of argument $x$ and order $n$, and $H_n(k_0 r)$ is the Hankel function of the second kind of order $n$ and argument $x$.

From equations (3), (5), (6) and (7) one notices that the scattered field from a wedge or a cylinder due to either plane wave or line-source field excitation has the appearance of a cylindrical wave emanating from a fictitious line source at the edge of the wedge or at the axis of the cylinder.

Consider now the case of two conducting wedges separated by a distance $2a$, where $2a > 1$ and a circular cylinder of radius $a$ whose axis is parallel to the edges of the two wedges, and where all three bodies are illuminated by a plane wave of unit amplitude (see Fig. 1). The field at any point is considered to be composed of the incident field plus a response field from each of the two wedges and the cylinder. The latter field consists of an unperturbed scattered field by the three scatterers due to the original plane wave plus an interaction field which will be represented by three fictitious line sources located at the wedge edges ($\varepsilon_e$ and $\varepsilon_d$) and at the cylinder axis. If the plane wave incidence angle is restricted such that the incident field does not illuminate the lower faces of the wedges, the total field $E'$ in the forward direction is given by

$$
E' = E^I + E^S
$$

where

$$
E^S = E^{S31} + E^{S32} + E^{S33}
$$

while

$$
E^{S31} = \frac{2}{d} H_3(k_0 r) \left[ \exp(-j k_0 r) \right] \cdot \left[ c_3, f_3(\phi_1, \phi_2, \phi_3, r_1) + c_3, f_3(\phi_1, \phi_2, \phi_3, r_1) \right]
$$

$$
E^{S32} = \frac{2}{d} H_3(k_0 r) \left[ \exp(-j k_0 r) \right] \cdot \left[ c_3, f_3(\phi_1, \phi_2, \phi_3, r_2) + c_3, f_3(\phi_1, \phi_2, \phi_3, r_2) \right]
$$

$$
E^{S33} = \frac{2}{d} H_3(k_0 r) \left[ \exp(-j k_0 r) \right] \cdot \left[ c_3, f_3(\phi_1, \phi_2, \phi_3, a) + c_3, f_3(\phi_1, \phi_2, \phi_3, a) \right]
$$

$$
r_1 = 2a - 2a
$$

$$
r_2 = 2a - 2a
$$

$$
r_3 = 2a - 2a
$$
and $c_1$, $c_2$, and $c_3$ are the unknown strengths of the line sources at $e_4$, $e_5$, and along the cylinder axis, respectively. If $\theta^2$ is expressed in a normalized coordinate system, where the $z$ axis coincides with the centre of the aperture as shown in Fig. 1 and the well-known far-field conditions are used, we have $\theta_0 = \theta_2 = \theta_3 = \pi/2 + \theta_0$, $\phi_0 = \pi$, $\phi_2 = \phi_3 = \pi$, $\psi_1 = \phi_1 = \phi_2 = \phi_3 = 3\pi/2 + \theta_0$, $\psi_2 = \phi_2 = \psi_3 = \pi + \theta_0$, $\psi_1 = \phi_1 = \psi_3 = 3\pi/2 + \theta_0$, $\psi_2 = \phi_2 = \psi_3 = \pi + \theta_0$, $\psi_1 = \phi_1 = \psi_2 = \phi_2$, and $s_4$ and $s_5$ are the distances between $e_4$ and $e_5$ and the cylinder axis, respectively.

For the determination of $c_1$, $c_2$, and $c_3$ one may follow the analysis of Karp and Russek (1956) by imposing the requirement that the fields scattered by the two wedges and the cylinder be consistent with one another. This leads to the following

$$2c_1 - c_3[F(\phi_2, \phi_3, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8, \psi_9, \psi_{10}, \psi_{11}, \psi_{12})] = \exp(-j\kappa \sin \theta_0)[F(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8, \phi_9, \phi_{10}, \phi_{11}, \phi_{12})]$$

where $\kappa$ is the wavenumber and $\theta_0$ is the angle of incidence. The transmission coefficient $T$ for plane wave incidence is calculated using the definition of Karp and Russek (1956, equation 31), namely

$$T = Re \left\{ (1 - j)\tilde{T} \right\} / 2k$$

where $\tilde{T}$ is $F(0, s, d, e_1, e_2, e_3)$ in the limit as $\theta_0$ approaches 0. It is obvious that as $a$ approaches zero $T$ approaches $T_e$, which is the transmission coefficient of a double wide wedge of aperture 2s given elsewhere (Elsnerbeni and Hamid 1984).

3. Accuracy of the solution
The error and the region of validity of the present solution are investigated by comparing with numerical results due to the exact solution of the unconfined slit given by Skavlen (1952) for a normally incident plane wave. Skavlen employed the method of separation of variables to the wave equation in an elliptic coordinate system and was able to calculate the transmission coefficient for slits with $0 < kx < 10$, correct to five decimal places. As shown in the Table our results are in good agreement with the exact values of $T$ for $kx > 2$, with the error not exceeding
4. Discussion of the results

The transmission coefficient for a slit with a circular cylinder at or below the centre of the aperture plane is shown in Figs. 2 and 3, respectively, for a normally incident plane wave (θ_0 = 0°) with k_0 = 0.5 and ε_2 = 4. When the dielectric cylinder is at the centre of the aperture plane, \( \bar{T} \) is always larger than \( T \). However, \( \bar{T} \) oscillates with decreasing amplitude for increasing \( k_s \) as expected and tends to unity as \( k_s \) tends to infinity in accordance with the geometrical optics value of \( \bar{T} \). However, for a conducting cylinder, \( \bar{T} \) is in general less than \( T \) if the same cylinder is shifted below the centre of the aperture plane (Fig. 3). \( \bar{T} \) in the presence of a dielectric cylinder oscillates around \( T \), whereas \( \bar{T} \) in the presence of a conducting cylinder oscillates with increasing amplitude and tends to unity as \( k_s \) tends to infinity. Figure 4 shows the diffraction patterns of an unloaded slit and the corresponding diffraction patterns in the presence of a dielectric cylinder for two different values of \( kd \) with

<table>
<thead>
<tr>
<th>( k_s )</th>
<th>Exact</th>
<th>Present</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.54430</td>
<td>0.55958</td>
<td>2.56</td>
</tr>
<tr>
<td>1.2</td>
<td>0.57693</td>
<td>0.57227</td>
<td>0.73</td>
</tr>
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<td>1.4</td>
<td>1.11119</td>
<td>1.16314</td>
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</tr>
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<td>1.6</td>
<td>1.26169</td>
<td>1.24654</td>
<td>1.21</td>
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<td>1.8</td>
<td>1.21239</td>
<td>1.24640</td>
<td>1.21</td>
</tr>
<tr>
<td>2.0</td>
<td>1.18426</td>
<td>1.18850</td>
<td>0.35</td>
</tr>
<tr>
<td>2.4</td>
<td>1.05650</td>
<td>1.08358</td>
<td>-0.26</td>
</tr>
<tr>
<td>3.0</td>
<td>0.97020</td>
<td>0.96819</td>
<td>-0.21</td>
</tr>
<tr>
<td>4.0</td>
<td>0.92524</td>
<td>0.92629</td>
<td>-0.10</td>
</tr>
<tr>
<td>5.0</td>
<td>1.04992</td>
<td>1.05197</td>
<td>0.20</td>
</tr>
<tr>
<td>6.0</td>
<td>0.95559</td>
<td>0.95575</td>
<td>0.02</td>
</tr>
<tr>
<td>7.0</td>
<td>0.97174</td>
<td>0.97317</td>
<td>0.14</td>
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<tr>
<td>8.0</td>
<td>1.03352</td>
<td>1.02477</td>
<td>0.88</td>
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<tr>
<td>9.0</td>
<td>1.00199</td>
<td>1.00276</td>
<td>0.08</td>
</tr>
<tr>
<td>10.0</td>
<td>0.98224</td>
<td>0.98342</td>
<td>0.12</td>
</tr>
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</table>

Slit transmission coefficients.

0.4\%. Also, for oblique incidence, our results agree with the curves provided by Millar (1958) for different values of \( \theta_0 \) away from grazing incidence (\( \theta_0 = 90° \)). Millar's investigation of the diffraction of an E-polarized plane wave was based on the solution by successive substitutions of a pair of integral equations. The resulting expression for \( \bar{T} \) was in the form of a series in inverse powers of \( k_s \) and in good agreement with the exact results for \( k_s \gg 4 \). This comparison emphasizes the validity of our solution for the unloaded slit for all sizes.

In the presence of the cylinder, there are no independent solutions for this configuration to compare with. However, if \( k_{1,2} > k_s \) or \( k_s \alpha \), the asymptotic far-field expression for the field scattered by the cylinder is valid for calculating the near-field around the cylinder (Hongo et al. 1977). Hence, when the cylinder is present the accuracy of the results is dependent on \( k_{1,2}, k_s, k_\mu \) and \( \epsilon_2 \). Also, it seems reasonable to concentrate on the numerical results of \( \bar{T} \) whenever \( k_s \gg 3 \) to avoid the less accurate results.
Figure 2. Slit transmission coefficient versus ka for $\theta_h = 0^\circ$ and $ka = 0.5$.

Figure 3. Slit transmission coefficient versus ka for $\theta_h = 0^\circ$, $ka = 0.5$ and $ka = 5$. 
Figure 4. Slit diffraction pattern versus angle $\theta$ for $\theta_0 = 0^\circ$, $ks = 8$ and $ka = 0.5$.

$ks = 8$, $ka = 0.5$ and $\epsilon_r = 4$. In the figure, the solid curve is the unloaded slit diffraction pattern, in good agreement with the corresponding pattern given by Keller (1957). The beamwidth for the case of $kd = 0$ is less than the beamwidth of the unloaded slit. This behaviour agrees with the increase of $T$ over $T'$ in Fig. 2 for $ks = 8$. On the other hand, for the case of $kd = 5$ the beamwidth is larger than that of the unloaded slit, and this yields a lower $T$ as shown in Fig. 3 for $ks = 8$.

To illustrate further the effect of $\epsilon_r$ on $T$, we present Figs. 5 and 6, where $\epsilon_r = 2$, $5$, $8$, $ka = 0.5$ and $\theta_0 = 0^\circ$. Figure 5 shows that as $\epsilon_r$ increases, $T'$ increases over $T'$ for all electrical separations ($ks$), whereas when $kd = 5$, Fig. 6 indicates that as $\epsilon_r$ increases the peak-to-peak values of the oscillations increase and the oscillations are always around unity.

The behaviour of $T'$ for an obliquely incident plane wave is shown in Figs. 7-10 for $\theta_0 = 20^\circ$ and $40^\circ$. For $kd = 0$, $\epsilon_r = 5$, $9$ and $ka = 0.3$, Fig. 7 shows that $T'$ can be higher than unity at a different value of $ks$ which corresponds to the unloaded slit case ($ka = 0$). Also it is observed that as $\epsilon_r$ increases, $T'$ decreases in the lower range of $ks$ ($ks < 6$). For the same cylinder parameters, but with $\theta_0 = 40^\circ$ (Fig. 9), the peak value of $T'$ lies in the lower range of $ks$ and increases with $\epsilon_r$, contrary to the
Figure 5. Slit transmission coefficient versus $ks$ for $\theta_0 = 0^\circ$ and $ka = 0.5$.

Figure 6. Slit transmission coefficient versus $ks$ for $\theta_0 = 0^\circ$, $ka = 0.5$ and $kd = 5$. 
Figure 7. Slit transmission coefficient versus $k S$ for $\theta_s = 20^\circ$.

Figure 8. Slit transmission coefficient versus $k S$ for $\theta_s = 20^\circ$ and $kd = 1.5$. 
Figure 9. Slit transmission coefficient versus $k_s$ for $\beta_s = 40^\circ$.

Figure 10. Slit transmission coefficient versus $k_s$ for $\beta_s = 40^\circ$ and $kd = 1.5$. 
Figure 11. Slit transmission coefficient versus $k_s$ for $\theta_s = 0^\circ$.

Figure 12. Slit transmission coefficient versus $k_s$ for $\theta_s = 0^\circ$ and $hd = 5$. 
behaviour for $\theta_b = 20^\circ$. This indicates that $\theta_b$ affects the peak location and value of $T$. When $kd = 1.5$ for the same cylinder parameters and with $\theta_b = 20^\circ$ (Fig. 8), or $\theta_b = 40^\circ$ (Fig. 10), one notices that the peak-to-peak value of oscillations is lower than the corresponding case when $kd = 0$. In Figs. 11 and 12, $\hat{T}$ is shown for a conducting cylinder with different values of $ka$, namely $0.1$, $0.2$, $0.8$, $kd = 0$ (Fig. 11), and $kd = 5$ (Fig. 12). The use of a conducting cylinder does not indicate any increase of $\hat{T}$ over $T$, and this may be due to the effect of blocking of part of the incident field by the cylinder.

The effect of the interior wedge angle on $T$ in the absence of the cylinder was previously investigated by Teague and Zitron (1972) and Elsheberbi and Hamid (1984), and similar results apply in the presence of the cylinder. However, as an example we present Fig. 13 where $\hat{T}$ of a double wedge in the presence of a conducting or dielectric cylinder is shown for $\theta = 0^\circ$, $ka = 0.5$, $\epsilon_r = 4$ and $\theta = 20^\circ$. In comparing Figs. 2 and 13, one notices that the interior wedge angle changes the levels of maxima and minima of the oscillations of $T$, whereas the peak position are the same.

The possibility of increasing or decreasing $\hat{T}$ over a certain range of $ka$ by using other arrangements (e.g. a dielectric cylinder or non-circular cross-section, or a lossy dielectric shell or cylinder, or a dielectric shell with inhomogeneous permittivity profiles (Elsheberbi and Hamid 1985b), or introducing a dielectric lens or several cylinders) is under investigation.

![Figure 13. Transmission coefficient versus ka of a double wedge for $\theta_b = 0^\circ$, ka = 0.5 and $\theta = 20^\circ$.](image-url)
5. Conclusions

Our solution for the scattering of an incident plane wave by three objects of different geometries and materials proved to be useful in determining the transmission coefficient through the aperture of a double wide wedge with a dielectric or conducting cylinder between or near the two wedge edges. The method of analysis is applicable to symmetric and asymmetric geometry and the generalization for a line-source excitation is simple and straightforward. The present geometry is an extension to the loaded slit treated by Hurdl and Sachdeva. The numerical results indicate that the use of a dielectric cylinder improves the transmission coefficient in general whereas the conducting cylinder acts as a blocking object resulting in a lower transmission coefficient.

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