Eigenvalues of propagating waves in a circular waveguide with an impedance wall

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Abstract: The paper determines the eigenvalues and conditions of propagation of hybrid modes inside a circular cylindrical waveguide with an impedance wall. The analysis allows for different longitudinal and transverse impedances on the walls. A general characteristic equation is derived from which the corresponding characteristic equations for circular cylindrical waveguides with a perfectly conducting wall, a corrugated wall or a dielectric lined wall can be obtained. Finally, it is found that the propagation of hybrid modes in circular waveguides with large radii is independent of the surface impedance.

1 Introduction

The determination of the propagating electromagnetic modes inside waveguides with an impedance wall has received considerable attention due to its importance in the development of new microwave and millimeter-wave components. The boundary value solution for the electromagnetic fields inside a circular waveguide using the surface impedance concept has been successfully used by several authors [1–4]. The use of this concept greatly simplifies the solution and allows the fields in the waveguide to be expressed in simple form for easy comparison with other types of circular waveguides. On the other hand, previous investigators, using perturbation techniques, have found that hybrid modes can propagate in cylindrical waveguides [5, 6]. However, their analysis is only valid for wall impedances which are nearly zero. Dragan has analyzed circular cylindrical waveguide and established the existence of hybrid modes for nonzero wall impedances [3]. The same conclusion is reached in the paper using a more basic approach and the concept of surface impedance is extended to several types of cylindrical structures including smooth-walled, corrugated, and impedance-walled waveguides.

2 Formulation

Using the cylindrical coordinates (r, ϕ, z) and assuming an e⁻ᵣt time dependence, the fields in a circular cylindrical waveguide can be written as a combination of TE and TM components in the form [7]

\[ E_z = (k^2 - k_{qu}^2)E_z \]  
\[ E_\phi = \frac{k}{r} \left( k_{qu} \left( \frac{\partial E_z}{\partial r} \right) \right) \]  
\[ H_z = (k^2 - k_{qu}^2)H_z \]  
\[ H_\phi = \frac{k}{r} \left( k_{qu} \left( \frac{\partial H_z}{\partial r} \right) \right) \]

where \( k \) and \( k_{qu} \) are the scalar electric and magnetic potentials, \( \mu_0 \) and \( \varepsilon_0 \) are the permeability and permittivity of free space, respectively, \( \omega \) is the angular frequency, \( k \) the propagation constant in the \( z \) direction and \( k \) the free space wavenumber.

For a circular cylindrical waveguide with radius \( a \) the boundary conditions on the inner wall are given by [9]

\[ E_z = -\eta_1 \eta_0 H_\phi, \quad r = a \]  
\[ E_\phi = \eta_0 \mu_0 H_z, \quad r = a \]

where \( \eta_1 \) and \( \eta_0 \) are the relative wall impedances in the \( r \) and \( \phi \) directions, respectively, and \( \eta_0 \) is the intrinsic impedance of free space. Substituting the field components into Eqs. 5 and 6 leads to

\[ (k^2 - k_{qu}^2)\psi_z = -\eta_1 \eta_0 \frac{j\omega \mu_0}{\eta_0} \frac{\partial \psi_z}{\partial r} \]  
\[ + \frac{k}{r} \left( k_{qu} \frac{\partial \psi_z}{\partial r} \right), \quad r = a \]  
\[ (k^2 - k_{qu}^2)\psi_\phi = -\frac{1}{\eta_0 \mu_0} \left[ \frac{k}{r} \left( k_{qu} \frac{\partial \psi_\phi}{\partial r} \right) - j\omega \varepsilon_0 \psi_\phi \right], \quad r = a \]

If the scalar potentials are defined by

\[ \psi_z = a_0 \cos (n\phi) \left[ 1 - k^2 - k_{qu}^2 \right]^{1/2} e^{j\omega t} \]  
\[ \psi_\phi = b_0 \sin (n\phi) \left[ 1 - k^2 - k_{qu}^2 \right]^{1/2} e^{j\omega t} \]

Eqs. 7 and 8 yield the following transcendental equation

\[ [\varepsilon_0 \mu_0 J_n(k_{qu} a) + \eta_1 \eta_0 k_n \mu_0 J_n(k a)] \times \left[ \begin{array}{r} J_{n+1}(k_{qu} a) + \frac{1}{\eta_0 \mu_0} k_{qu} \mu_0 J_{n+1}(k a) \\ J_{n+1}(k a) \end{array} \right] + \frac{\eta_1}{\eta_0} n^2 \left[ k^2 - k_{qu}^2 - \varepsilon_0 \mu_0 J_n^2(k_{qu} a) \right] = 0 \]  

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where $J_1(x)$ is the Bessel function of the first kind, of order $n$ and argument $x$, and the prime denotes differentiation with respect to the argument. The order of the zeros of $J_1(x)$ in Eqn. 11 are denoted by the subscript $p$, whereas the eigenvalues are given by

$$\nu_n^2 = \left( p^2 + \frac{1}{4} \right)^2 a_n^2$$

(12)

Special, and at the same time practical, cases of surface impedances can easily be analysed using Eqn. 11. As an example, for an isotropic wall impedance ($\eta_\perp = \eta_\parallel = \eta$) we have

$$\nu_n^2 = J_2(a_n) + \sqrt{\frac{\eta a_n}{\eta_\perp}} J_1(a_n)$$

(13)

which is identical to Eqn. 1 of Reference 1. Furthermore, for a perfectly conducting wall ($\nu_n = 0$) Eqn. 11 reduces to

$$\nu_n^2 = J_2(a_n) - \sqrt{\frac{\eta a_n}{\eta_\perp}} J_1(a_n) = 0$$

(14)

therefore

$$J_2(a_n) = 0 \ \text{or} \ \eta_\perp = \eta$$

(15)

which are the well known characteristic equations for the $TE$ and $TM$ modes in a circular cylindrical waveguide with a perfectly conducting wall.

Another important geometry is a corrugated waveguide with one-quarter wavelength slots which can be simulated by setting $\eta_\parallel = \infty$ and $\eta_\perp = 0$ into Eqn. 11. This yields

$$\nu_n^2 = \frac{1}{4} J_2(a_n) - \sqrt{\frac{\eta a_n}{\eta_\parallel}} J_1(a_n) = 0$$

(16)

which are the characteristic equations for the unbalanced hybrid modes in a circular cylindrical waveguide [9]. Perhaps the most interesting case is for large $ka$ and imperfectly conducting walls provided that neither $\eta_\perp$ nor $\eta_\parallel$ is zero [3]. Under these boundary conditions, Eqn. 11 reduces to

$$\nu_n^2 = \frac{1}{4} J_2(a_n) + \sqrt{\frac{\eta a_n}{\eta_\parallel}} J_1(a_n) = 0$$

(17)

which are the characteristic equations for balanced $HE$ and $EH$ modes in a circular cylindrical waveguide [10].

3 Results and conclusions

The eigenvalues are determined numerically using the ZXSSQ minimisation routine of the International Mathematical and Statistical Library (IMSL). The resulting $\nu_n^2$ are found to be accurate up to at least 5 decimal places when compared with the tabulated eigenvalues for the special cases of a perfectly conducting wall and a corrugated wall under balanced hybrid conditions.

For an isotropic ($\eta_\perp = \eta_\parallel = \eta$) as well as perfectly absorbing wall ($\eta = 1$) [11] complex values of $\nu_n$ are shown in Table 1 (for $ka = 5$) which signifies attenuation of a $z$ directed wave. Table 2 lists $\nu_n$ of some of the propagating modes inside a cylindrical waveguide with a lossless wall impedance for $ka = 10, \eta_\perp = 2.5$ and $\eta_\parallel = 0.4$. All of these results shown in this table are real which indicates that these hybrid modes will propagate without attenuation; however, these modes are not necessarily balanced hybrid modes.

Fig. 1 displays $\nu_n$ for the fundamental mode ($HE_{11}$) against $ka$ based on Eqn. 11 for various values of wall impedance. In all of these cases the wall impedances in the two tangential directions are related by the expression $\eta_\perp = \eta_\parallel$, and it was found that if $\eta_\perp = 0.4583$ and $\eta_\parallel = 2.40483$ then $\nu_n = 2.40483$ for all values of $ka$

### Table 1: $\nu_n$ for $ka = 5$ and $\eta_\perp = \eta_\parallel = 1.0$

<table>
<thead>
<tr>
<th>$\nu_n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_n$</td>
<td>3.487</td>
<td>-0.731</td>
<td>2.277</td>
<td>-0.490</td>
<td>3.573</td>
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<tr>
<td>4.894</td>
<td>-0.831</td>
<td>5.458</td>
<td>-1.541</td>
<td>8.797</td>
<td>-0.389</td>
</tr>
<tr>
<td>5.390</td>
<td>-0.943</td>
<td>4.506</td>
<td>-0.851</td>
<td>4.506</td>
<td>-0.960</td>
</tr>
<tr>
<td>7.867</td>
<td>-0.635</td>
<td>4.789</td>
<td>-1.044</td>
<td>4.872</td>
<td>-0.535</td>
</tr>
<tr>
<td>6.907</td>
<td>-0.783</td>
<td>8.156</td>
<td>-1.155</td>
<td>12.348</td>
<td>-0.303</td>
</tr>
<tr>
<td>8.123</td>
<td>-0.440</td>
<td>7.154</td>
<td>-0.710</td>
<td>8.656</td>
<td>-0.831</td>
</tr>
<tr>
<td>13.027</td>
<td>-0.327</td>
<td>11.183</td>
<td>-0.743</td>
<td>12.639</td>
<td>-0.711</td>
</tr>
<tr>
<td>14.944</td>
<td>-0.347</td>
<td>7.278</td>
<td>-0.891</td>
<td>11.670</td>
<td>-0.551</td>
</tr>
<tr>
<td>13.067</td>
<td>-0.524</td>
<td>14.380</td>
<td>-0.282</td>
<td>15.732</td>
<td>-0.485</td>
</tr>
<tr>
<td>18.078</td>
<td>-0.284</td>
<td>10.237</td>
<td>-0.586</td>
<td>14.816</td>
<td>-0.406</td>
</tr>
<tr>
<td>16.241</td>
<td>-0.391</td>
<td>14.408</td>
<td>-0.503</td>
<td>18.994</td>
<td>-0.287</td>
</tr>
<tr>
<td>21.216</td>
<td>-0.240</td>
<td>13.347</td>
<td>-0.425</td>
<td>17.970</td>
<td>-0.322</td>
</tr>
<tr>
<td>19.418</td>
<td>-0.312</td>
<td>17.631</td>
<td>-0.378</td>
<td>22.225</td>
<td>-0.296</td>
</tr>
<tr>
<td>24.359</td>
<td>-0.208</td>
<td>18.482</td>
<td>-0.333</td>
<td>21.123</td>
<td>-0.267</td>
</tr>
<tr>
<td>22.588</td>
<td>-0.290</td>
<td>20.835</td>
<td>-0.303</td>
<td>28.629</td>
<td>-0.214</td>
</tr>
</tbody>
</table>

### Table 2: $\nu_n$ for $ka = 10, \eta_\perp = 2.5, \eta_\parallel = 0.4$

<table>
<thead>
<tr>
<th>$\nu_n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_n$</td>
<td>3.993</td>
<td>2.404</td>
<td>3.815</td>
<td>5.092</td>
<td>6.296</td>
</tr>
<tr>
<td>17.881</td>
<td>18.596</td>
<td>18.652</td>
<td>20.314</td>
<td>21.720</td>
<td></td>
</tr>
</tbody>
</table>

[Fig. 1] $\nu_n$ against $ka$ for various values of $\eta_\parallel$
above cutoff (\(ka = 1.3569\), which is the smallest allowable value of \(ka\) for the \(TE_{11}\) mode to propagate in a perfectly conducting waveguide). This eigenvalue signifies the balanced \(HE_{11}\) mode and shows that balanced hybrid conditions can be established in cylindrical waveguides with a small \(ka\) and reactive wall impedances. This requirement may be satisfied in practice with corrugated metal surfaces.

The asymptotic behavior of \(u_{11}\) of the \(HE_{11}\) mode as \(ka\) increases for real wall impedance, evaluated using Eqn. 13 is shown in Fig. 2. As can be seen from the figure, the real part of \(u_{11}\) approaches 2.4048 (which is the eigenvalue of the balanced \(HE_{11}\) mode) while the imaginary part approaches zero. This result is supported by Dragone which predicts the propagation of hybrid modes for waveguides with impedance walls and only for large \(ka\) [3]. This result is also verified by Eqn. 16 which shows that for large values of \(ka\) the balanced hybrid mode propagates for any nonzero value of wall impedance. Furthermore, similar behavior was reported by Clarricoats and Taylor for the dielectric loaded circular waveguide [12]. The one exception to this is if both tangential impedances are equal and approach infinity. This case is shown in Fig. 3 where \(u_{11}\) of the \(HE_{11}\) mode approaches the \(TE_{11}\) eigenvalue for large \(\eta\) and for several values of \(ka\) [13]. Fig. 4 shows the corresponding behavior of \(k_{1}\) (for the same values of \(ka\) as in Fig. 3) as a function of \(\eta\). Again, it is obvious that the imaginary part of \(k_{1}\), which accounts for the attenuation of the propagating wave, approaches zero for large values of \(ka\) and all values of \(\eta\) greater than zero.

While the hybridity effect can be achieved by a lossless dielectric lining on the waveguide wall, there is, never the less, an advantage in using impedance type (or lossy dielectric) lining such that the lining will absorb the


Fig. 2 Real and imaginary parts of \(u_{11}\) against \(ka\) for \(\eta = 1.0\)

Fig. 3 Real and imaginary parts of \(u_{11}\) against \(\eta\)

Fig. 4 Real and imaginary part of \(k_{1}\) against \(\eta\) for \(\eta = 1.0\)
undesirable $EH$ surface modes at a reasonable increase in propagation losses. The extension to aperture type antennas or feed elements with impedance wall linings is obvious in the sense that the reduction in the sidelobe level (or radiation pattern envelope) can be achieved at a small drop in gain over a reasonably wide frequency band (due to the frequency dependence of wall impedances).

4 Acknowledgments

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5 References

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