Scattering by a perfectly conducting strip loaded with a dielectric cylinder (TM case)

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Abstract: A rigorous analysis of the problem of scattering from an infinite strip of semi-conductor material is presented for a plane and plane wave excitation. The vector Green's theorem is used to derive an integral equation for the unknown electric and magnetic field components. The singular component of the magnetic field is then cancelled out by properly choosing the auxiliary fields and boundary conditions. The unknown expansion coefficients of the scattered field are determined by numerically solving an infinite set of simultaneous linear equations after truncation. The accuracy of the proposed solution is verified for the loaded and unloaded strip by comparing the results of this technique with those based on the method of moments. For the loaded strip, the dependence of the scattered field on the angle of incidence is investigated and presented for plane wave incidence.

1 Introduction

The scattering by a thin perfectly conducting strip has been studied by many authors using different techniques. The exact solution of a plane wave scattered by a conducting strip was first given by Morse and Rubenstein in terms of Mathieu functions by solving the Helmholtz equation in elliptical cylindrical co-ordinates [1]. Approximate solutions for the low and high frequencies were also investigated [2-7]. Previous attempts to maximize or minimize the scattering cross-section of thin strips include the use of resistive rather than conducting materials [8, 9] and loading the edges by cylindrical dielectric shells of homogeneous or inhomogeneous permittivity profiles [10]. This paper examines the scattering properties of a thin perfectly conducting strip loaded by a dielectric cylinder as shown in Fig. 1. The dielectric medium is assumed to be linear, homogeneous, isotropic and free from losses and characterized by relative permittivity \( \varepsilon_r \). The analysis is based upon integral equations derived from Green's second identity. The reciprocity theorem is used along with appropriate boundary condi-

2 Basic formulation

Consider the E-polarisation case (TM with respect to the z-axis) where the electric field has two components only of the circular cylindrical co-ordinates (\( \rho, \phi, z \)). A time dependence \( e^{j\omega t} \) is assumed and suppressed throughout. Furthermore, consider two Maxwellian fields \((E, H)\) and \((E_0, H_0)\) of the same frequency, supported, respectively, by sources \((J, M)\) and \((J_0, M_0)\) which might exist within a surface \( S \) enclosing a volume \( V \) of a linear, homogeneous and isotropic medium. An application of Green's second identity, or Green's vector theorem yields [11]

\[
\int_V \left( E_x \times H - E \times H_0 \right) \cdot dS = \int_S \left[ (E \cdot J - E_0 \cdot J_0) - (H_0 \cdot M + H \cdot M_0) \right] dV
\]

where \( h \) is a unit normal directed out of \( V \). Let \( E \) and \( H \) represent the unknown fields due to the sources \( J \) and \( M \), while \( E_0 \) and \( H_0 \) are the chosen fields, the sources of which will always be situated outside the region of application. To obtain a solution for the real fields, we divide the space into three regions where the integer \( m \) denotes the region number. Regions I and II are inside the dielec-

tric cylinder and are separated by the conducting strip.
whereas region III is outside the cylinder. We also assume that the strip lies in the \(z\)-\(x\) plane, the axis of the cylinder coincides with the \(z\)-axis and the radius of the cylinder is equal to half the width of the strip (denoted by \(a\)) as shown in Fig. 1. It is convenient to make a different choice of \(E_a\) and \(H_a\) for the different regions, though in all regions they are source free, i.e., \(J_x\) and \(M_x\) are zero. If the real sources \(J\) and \(M\) are in region III and with \(M = 0\), an application of eqn. 1 to region I yields

\[
\int_{0}^{2\pi} (E_{1z}, H_{10} - E_{10}, H_{1z}) \, d\phi = \mathbf{0} \quad p = a
\]  

(2)

similarly for region II, we have

\[
\int_{0}^{2\pi} (E_{2z}, H_{20} - E_{20}, H_{2z}) \, d\phi = \mathbf{0} \quad p = a
\]  

(3)

while for region III, eqn. 1 reduces to

\[
\int_{0}^{2\pi} (E_{3z}, H_{30} - E_{30}, H_{3z}) \, d\phi = -\frac{1}{a} \epsilon(E_{3}\phi_{0}, \phi_{a}) \quad p = a
\]  

(4)

Since there is no \(z\) variation in either region, the three regions are assumed to be of arbitrary finite length in the \(z\) direction. As a result, the integration with respect to \(z\) on both sides of eqn. 1 results in the assumed finite length which is then cancelled out from both sides. The surface integrals over the end caps of a cylindrical region also cancels each other since the outward normals are oppositely directed. Also, the fields \(E_{1}\), \(E_{2}\), and \(E_{3}\) are assumed to be zero on the strip surface (to satisfy the Dirichlet boundary condition) while the far field radiation condition is applied. In eqn. 4 \(J_d\) is the intensity of the electric line source located at \(\phi_{0}, \phi_{a}\). Eqns. 2-4 represent three integral equations to be solved for the real electric and magnetic fields \(E_{j}\) and \(H_{j}\) on the boundary \(p = a\). However, due to the singularity of the magnetic field at the edge of the conducting strip, it is better to eliminate the unknown magnetic field from these equations, leaving the finite electric field to be determined. In order to extract \(H_0\) one may choose for convenience the following boundary conditions:

\[
E_{1} = E_{2}, \quad p = a, \quad 0 \leq \phi < \pi \]

(5)

\[
E_{2} = E_{3}, \quad p = a, \quad 0 < \phi \leq 2\pi \]

(6)

Hence by substituting eqns. 2 and 3 from eqn. 4 and using the relations given by eqns. 5 and 6, we obtain

\[
\int_{0}^{2\pi} E_{1z}, H_{10} \, d\phi + \int_{0}^{2\pi} E_{10}, H_{1z} \, d\phi = \int_{0}^{2\pi} E_{2z}, H_{20} \, d\phi - \int_{0}^{2\pi} E_{20}, H_{2z} \, d\phi - \frac{1}{a} \epsilon(E_{3}\phi_{0}, \phi_{a}) \quad p = a
\]  

(7)

where the integrals containing the unknown \(H_0\) cancel out.

The auxiliary fields \(E_{1z}\), \(E_{2z}\) may be expanded as Fourier series equated to \(E_{3z}\) over the relevant ranges. Hence, we assume that

\[
E_{1z} = \sum_{n} A_{n}(\phi) \cos(n\phi + \nu) \sin(n\phi + \nu)
\]  

(8)

\[
E_{2z} = \sum_{n} B_{n}(\phi) \cos(n\phi + \nu) \sin(n\phi + \nu)
\]  

(9)

where \(E_{3z} = H_{3}\phi_{0}\) may take the following form

\[
E_{3z} = H_{3}\phi_{0} \sin\phi \quad \rho = 0, \pm 1, \pm 2, \ldots
\]  

(10)

In the above equations \(J_{d}(\phi)\) is the Beusen function of the first kind of order \(n\) and argument \(\alpha\), \(k\) is the wave number in free space, \(k = \sqrt{\mu_0\epsilon_0} k\). \(H_{d}(\phi)\) is the Hankel function of order \(n\) and argument \(\alpha\), while the superscript (2) is implied and suppressed throughout. It should be noted that the expansion coefficients \(A_n\) and \(B_n\) are determined by imposing the boundary conditions given by eqns. 5 and 6 which yield

\[
A_{n}(\phi) = \frac{2H_{n}(\alpha\phi)}{\sqrt{T_{n}(\alpha\phi)}} \int_{0}^{2\pi} \epsilon(E_{3}\phi_{0}, \phi_{a}) \sin(n\phi + \nu) \, d\phi
\]  

(11)

\[
B_{n}(\phi) = \frac{2H_{-n}(\alpha\phi)}{\sqrt{T_{n}(\alpha\phi)}} \int_{0}^{2\pi} \epsilon(E_{3}\phi_{0}, \phi_{a}) \sin(n\phi + \nu) \, d\phi
\]  

(12)

Furthermore, the corresponding \(\phi\) components of the magnetic fields are

\[
H_{d}(\phi) = \frac{1}{\alpha} \sum_{n} A_{n}(\phi) \cos(n\phi + \nu) \sin(n\phi + \nu)
\]  

(13)

\[
B_{n}(\phi) = \frac{1}{\alpha} \sum_{n} B_{n}(\phi) \sin(n\phi + \nu)
\]  

(14)

\[
H_{d}(\phi) = \frac{1}{\alpha} \sum_{n} H_{n}(\phi) \phi_{0}\sin(n\phi + \nu)
\]  

(15)

where the primes denote differentiation with respect to the full argument \(\phi\) and \(\chi\) is the intrinsic impedance of free space and \(\mu_{0} = \chi_{0}\). Now, substitution of the auxiliary field expression into eqn. 7 yields

\[
\frac{1}{\pi} \sum_{n} \sum_{m} A_{n}(\phi) \cos(n\phi + \nu) \sin(m\phi + \nu) \int_{0}^{2\pi} E_{3z, n, m} \, d\phi
\]

\[
+ \frac{1}{\pi} \sum_{n} \sum_{m} B_{n}(\phi) \sin(n\phi + \nu) \int_{0}^{2\pi} E_{3z, n, m} \, d\phi
\]

\[= \frac{1}{\alpha} \sum_{n} H_{n}(\phi) \phi_{0}\sin(n\phi + \nu)
\]  

(16)

The solution of the above equation gives the values of the field \(E_{3z}\) at the boundary \(p = a\). Assuming the following series for \(E_{j}\) in region III

\[
E_{j} = \sum_{n} \chi_{j, n}(\phi) \cos(n\phi + \nu) \sin(n\phi + \nu)
\]  

(17)

and substituting eqns. 17 into eqn. 16, we obtain

\[
\sum_{n} \chi_{j, n}(\phi) \cos(n\phi + \nu) \sin(n\phi + \nu) \int_{0}^{2\pi} E_{3z, n, m} \, d\phi
\]

\[= \frac{1}{\alpha} \sum_{n} H_{n}(\phi) \phi_{0}\sin(n\phi + \nu)
\]  

(18)

where \(C_{j}\) and \(C_{j}\) are the unknown coefficients and \(J_{d}\) and \(f_{d}\) are given by

\[
f_{d}(\phi, q) = \left[ m Q_{n}(\phi) - p_{n}(\phi) \right] \sin(q \phi)
\]

(19)

\[
\pm \frac{m}{2} \sin \phi_q + \pm \frac{p}{2} \sin \phi_P
\]

\[
\pm \frac{m}{2} \sin \phi_q + \pm \frac{p}{2} \sin \phi_P
\]  

(20)
Then the total field is given as

\[ E_z = \sum_{\nu = \pm \infty} e^{j \nu \phi} \left( E_\nu e^{j \nu \phi} k_{j} + d_\nu H_{j} \phi(k) \right) \]  

In comparing eqns. 17 and 29 at \( p = a \), the coefficient \( d_\nu \) reduces to

\[ d_\nu = C e^{-j \nu \phi} \frac{E_\nu e^{j \nu \phi} k_{j} \phi(k)}{H_{j} \phi(k)} \]  

The far scattered field pattern \( F(\phi) \) can be determined after using the large argument approximation of the Hankel function and normalizing the resulting expression by the factor \( \sqrt{2} k e^{-j \phi/2} \). Thus, the scattered field pattern due to a line source excitation \( F_\nu(\phi) \), and due to an incident plane wave \( F(\phi) \), are given, respectively, by

\[ F(\phi) = \frac{-mk}{4} \sum_{\nu} \frac{\sin \frac{\nu \phi}{2}}{\nu} \]  

The scattering cross-section of the loaded strip is then given by Reference 12 as

\[ \sigma(\phi) = \frac{1}{k} \int F(\phi)^2 \]  

4 Numerical results and discussion

Examination of our analytical solution indicates that as \( k_0 \) approaches infinity the right-hand side and the second term of the left-hand side of eqn. 21 reduce to zero, which results in a zero value for the coefficients \( C_\nu \). Thus, the coefficients \( d_\nu \) of the scattered field due to a line source excitation (eqn. 23) or an incident plane wave (eqn. 30) reduce to the well-known expressions for the scattering by a perfectly conducting cylinder. To further check the accuracy of the proposed solution, the scattered field pattern of loaded and unloaded conducting strip are calculated by using the well-known method of moments. As shown in Figs. 2 and 3 for \( d_\nu = 0^\circ \) and \( 90^\circ \), respectively, good agreement between the results based on the proposed technique and the method of moments is obtained for both \( k_0 = 1 \) and \( k_0 = 5 \). Fig. 4 shows the forward and backward scattering cross-sections (\( \sigma_1 \) and \( \sigma_2 \), respectively) versus \( N \) to check the convergence of the proposed solution for loaded strips. In this Figure, \( \sigma_1 = 9 \) and \( \sigma_2 = 90^\circ \) while the parameter \( k_0 \) is set equal to 1, 2, and 5. It can be seen that the computed values of \( \sigma_1 \) and \( \sigma_2 \) converge rapidly with the first few expansion coefficients which is attributed to the efficiency of the proposed solution. Fig. 5 shows the scattered field pattern of a loaded strip excited by a normally incident plane wave where \( k_0 = 1 \) and 5 in Figs. 5a and 5b, respectively, and \( \phi \) is a parameter. In Fig. 5a, the calculated values of the scattered fields, based on the method of moments are also shown for \( \phi = 5 \) and 9. Good agreement is observed between the numerical results based on the proposed method and the moment method, however one should point out that in the method of moment solution the orders of the impedance matrix were 100 and 130 for \( \psi = 5 \) and 9, respectively, while in the proposed method \( N \) was set to 18 for both cases. Although it is clear from Fig. 5b that the forward scattered field for \( k_0 = 5 \) increases with \( \phi \) and the back scattered field decreases with \( \phi \), it is not possible to have this behaviour in general, as shown in Fig. 5a for \( k_0 = 1 \).
To further investigate the behaviour of the forward and back scattered fields with respect to $\phi_1$, we present Fig. 6 which shows the changes of $\sigma_f$ and $\sigma_b$ versus $\phi_1$ for $\phi_0 = 90^\circ$, while the parameter $k a$ is equal to 1, 2 and 5. Fig. 7 also shows the variations of $\sigma_f$ and $\sigma_b$ versus $\phi_1$ for different values of $\phi_0$, namely $0^\circ$, $30^\circ$ and $60^\circ$ while $k a = 5$.

Fig. 2 Normalized scattered field patterns due to plane wave incident on conducting strip with $\phi_0 = 0^\circ$

- $\phi_0 = 0^\circ$
- $k a = 5$
- proposed solution
- method of moments

Fig. 3 Normalized scattered field patterns due to plane wave incident on conducting strip with $\phi_0 = 90^\circ$

- $\phi_0 = 90^\circ$
- $k a = 5$
- proposed solution
- method of moments

Fig. 4 Scattering cross-section of loaded strip due to plane wave incident with $\phi_0 = 90^\circ$ and $\phi_1 = 0$ against $N$

- Forward scattering cross-section at $\phi = 90^\circ$
- $k a = 1$
- $k a = 3$
- $k a = 5$

$ka = 2$. From Figs. 6 and 7 one can easily observe that the loaded strip resonates at different values of $e_r$, $e_0$, and $ka$ while the number of resonances increases with $e_r$. For the normally incident plane wave case. Finally, Fig. 8 shows the plots of $u_r$ and $a_n$ against $ka$ for $\phi_0 = 90^\circ$ and $e_r$ as a parameter which is equal to 1, 5 and 9. From the

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**Fig. 5** Far scattered field pattern due to plane wave incident on loaded strip with $\phi_0 = 90^\circ$

- $ka = 1$
- $ka = 3$
- $e_r = 1$
- $e_r = 5$
- Method of moments

**Fig. 6** Scattering cross-section of loaded strip due to plane wave incident with $\phi_0 = 90^\circ$ against relative permittivity $e_r$.

- Forward scattering cross-section at $\phi = 90^\circ$
- Back scattering cross-section at $\phi = 90^\circ$
- $ka = 1$
- $ka = 2$
- $ka = 3$

**Fig. 7** Scattering cross-section of loaded strip due to plane wave incident with $ka = 2$ against relative permittivity $e_r$.

- Forward scattering cross-section
- Back scattering cross-section
- $\phi_0 = 30^\circ$
- $\phi_0 = 60^\circ$

Figure one concludes that it is possible to increase $\sigma_f$ at the same time decrease $\sigma_a$ for all values of $k_0$, which are generally greater than 2 except at resonance, by increasing the value of $k_0$. From the above results it seems that the scattering cross-section of a conducting strip can be greatly minimised or maximised by loading it with a dielectric cylinder. However, may not be a simple relation between the optimal value of the scattering cross-section and the parameters $k_0$, $\sigma_f$, and $\sigma_a$.

5 References
