1. Consider a perfect fluid in a static, circularly symmetric (2+1)-dimensional spacetime, equivalently, a cylindrical configuration in (3+1)-dimensions with perfect rotational symmetry.
   a) Show that the vacuum solution can be written as
   \[ ds^2 = -dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\theta^2 \]
   where \( M \) is constant and \( \theta \in (0, 2\pi) \).
   b) Derive the analogue of the Tolman-Oppenheimer-Volkoff equation for (2+1)-dimensions.
   c) Solve the (2+1)-dimensional TOV equation for a constant density star. Find \( p(r) \) and solve for the metric.

   a) You can use Mathematica to verify that
   \[
   g_{\mu\nu} = \begin{pmatrix}
   -1 & 0 & 0 \\
   0 & \frac{1}{1 - \frac{2M}{r}} & 0 \\
   0 & 0 & r^2 \\
   \end{pmatrix}
   \]
   with \( \{ t, r, \theta \} \) satisfies \( R_{\mu\nu} = 0 \).

   b) Consider a 2+1 D circular geometry that is assumed to be static, i.e.,
   \[
   ds^2 = -\frac{2\alpha(r)}{r^2} dt^2 + \frac{2\alpha(r)}{r^2} dr^2 + r^2 d\theta^2
   \]
   Using this metric and Mathematica we obtain:
   \[
   G_{tt} = \frac{2\alpha(r)}{r^2} \frac{d\alpha}{dr}
   \]
   \[
   G_{rr} = \frac{1}{r} \frac{d\alpha}{dr}
   \]
   \[
   G_{\theta\theta} = -\frac{1}{r^2} \frac{d\alpha}{dr} \left[ \left( \frac{d\alpha}{dr} \right)^2 - \frac{d\alpha}{dr} \frac{d\theta}{dr} + \frac{d\theta}{dr} \frac{d\theta}{dr} \right]
   \]
   Assuming a perfect fluid in its overall rest frame we have:
   \[
   \mathbf{t}_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} + p g_{\mu\nu} = \text{diag} \left( e^{\frac{\alpha}{r}}, e^{\frac{\theta}{r}}, r^2 \rho \right)
   \]
   or
   \[
   \mathbf{t}_{\mu\nu} = \text{diag} \left( e^{-\frac{\alpha}{r}}, -\frac{\theta}{r}, r^2 \rho \right) \quad \text{Note } \rho, p \text{ depend on } r.
   \]
Then $C_{tt} = 8\pi G T_{tt}$ becomes:

$$\frac{1}{r} e^{-2\phi} \frac{d\lambda}{dr} = 8\pi G e \rho$$

$$e^{-2\phi} \frac{d\beta}{dr} = 8\pi G r \rho$$

And $C_{rr} = 8\pi G T_{rr}$ becomes:

$$\frac{1}{r} \frac{d\lambda}{dr} = 8\pi G e \rho$$

$$\frac{d\nu}{dr} = 8\pi G e^{-1} r \rho$$

Lastly we can use $\nabla_{\tau} T^{\tau\nu} = 0$ with $\nu = r$ to obtain:

$$\nabla_{\tau} T^{\tau\nu} = 0 = \nabla_{t} T^{tr} + \nabla_{r} T^{rr} + \nabla_{\theta} T^{\theta r}$$

$$= \partial_{t} T^{tr} + \Gamma^{t}_{\lambda\nu} T^{\lambda r} + \Gamma^{r}_{\lambda t} T^{\tau \lambda}$$

$$+ \partial_{r} T^{rr} + \Gamma^{r}_{\lambda \nu} T^{\lambda r} + \Gamma^{r}_{r \lambda} T^{\tau \lambda}$$

$$+ \partial_{\theta} T^{\theta r} + \Gamma^{\theta}_{\lambda \nu} T^{\lambda r} + \Gamma^{\theta}_{r \lambda} T^{\tau \lambda}$$

From above:

$$\Gamma^{r}_{rr} = \frac{d\lambda}{dr} \Gamma^{r}_{t \nu} = -e^{-2\phi} \Gamma^{r}_{t \nu} = \frac{d\lambda}{dr} e^{2\phi - \lambda} \Gamma^{r}_{t \nu} = \frac{1}{r} \Gamma^{t}_{tr} = \frac{d\lambda}{dr}$$

Using these and keeping only nonzero terms (remember $T^{\tau\nu}$ is diagonal):

$$\nabla_{\tau} T^{\tau\nu} = 2\Gamma^{r}_{rr} T^{rr} + \Gamma^{r}_{or} T^{rr} + \Gamma^{r}_{ro} T^{rr} + \Gamma^{r}_{tr} T^{rr} + \Gamma^{r}_{tr} T^{rr} + \Gamma^{r}_{rr} T^{tt}$$

$$= \frac{d\lambda}{dr} (e^{-2\lambda} \rho) + \frac{d\lambda}{dr} e^{-2\lambda} \rho + \frac{d\lambda}{dr} e^{-2\lambda} \rho = \frac{d\lambda}{dr} e^{-2\lambda} \rho + \frac{d\lambda}{dr} e^{-2\lambda} \rho + \frac{d\lambda}{dr} e^{-2\lambda} \rho$$

$$= 0$$
Simplifying:
\[ 0 = \frac{dp}{dr} + \frac{d\alpha}{dr} (\rho + p) \Rightarrow \frac{dp}{dr} = -\frac{d\alpha}{dr} (\rho + p) \]

Now we make the “inspired” replacement:
\[ e^{-\alpha(r)} = 1 - 8\pi \mathcal{M}(r) \Rightarrow \frac{d}{dr} (e^{-\alpha}) = -\frac{d\alpha}{dr} e^{-\alpha} \]
\[ \text{or } \frac{d\alpha}{dr} e^{-\alpha} = -\frac{1}{1 - 8\pi \mathcal{M}(r)} \frac{d\mathcal{M}}{dr} \]

Then \( G_{tt} = 8\pi G\mathcal{M}_{tt} \) becomes:
\[ 4\pi \frac{d\mathcal{M}}{dr} = 8\pi G r \rho \]
\[ \Rightarrow \frac{d\mathcal{M}}{dr} = 2\pi r \rho \Rightarrow \mathcal{M}(r) = 2\pi \int_0^r \rho(r') dr' \]

Combining all 3 eqns. and eliminating \( \alpha \) and \( \beta \) yields:
\[ 8\pi G r (1 - 8\pi \mathcal{M}(r))^{-1} \rho = -\frac{1}{(\rho + p)} \frac{dp}{dr} \]

or
\[ \frac{dp}{dr} = -\frac{(\rho + p) 8\pi G r \rho}{1 - 8\pi \mathcal{M}(r)} \]

\[ \text{The 2+1D T.O.U. eqn.} \]

(1) Assuming constant density, i.e. \( \rho(r) = \frac{\rho_x}{r} \) \( 0 \leq r \leq R \)

we have:
\[ \mathcal{M}(r) = 2\pi \int_0^r \rho_x(r) dr' = \pi r^2 \rho_x \quad r \leq R \]

then:
\[ \frac{dp}{dr} = -\frac{(\rho_x + p) 8\pi G r \rho_x}{1 - 8\pi \rho_x r^2} \Rightarrow \frac{dp}{dr} = -\frac{8\pi G r dr}{(\rho_x + p) r} \]
Integrating both sides:

\[- \frac{1}{\rho^*} \ln \left( \frac{\rho^*}{\rho} + 1 \right) = \frac{1}{2\rho^*} \ln \left( 1 - 8\pi G \rho^* r^2 \right) + C\]

or

\[\ln \left( \frac{\rho^*}{\rho} + 1 \right)^{-1} = \ln \left( 1 - 8\pi G \rho^* r^2 \right)^{-\frac{1}{2}} + C\]

\[\frac{\rho^*}{\rho} = \frac{1}{e^{\frac{C}{\sqrt{1 - 8\pi G \rho^* r^2}}} - 1}\]

\[P(r) = \frac{\rho^*}{\rho} = \frac{\rho^* e^{\frac{C}{\sqrt{1 - 8\pi G \rho^* r^2}}} - 1}{e^{\frac{C}{\sqrt{1 - 8\pi G \rho^* r^2}}} - 1} \]

In terms of \( M = \pi R^2 \rho^* \) this becomes:

\[P(r) = \frac{\rho^* e^{\frac{C}{\sqrt{1 - 8GM \left( \frac{r}{R} \right)^2}}} - 1}{e^{\frac{C}{\sqrt{1 - 8GM \left( \frac{r}{R} \right)^2}}} - 1}\]

Now we need to make sure that \( p(R) = 0 \) (the pressure should vanish outside of the source).

\[p(R) = \frac{\rho^* e^{\frac{C}{\sqrt{1 - 8GM}}} - 1}{e^{\frac{C}{\sqrt{1 - 8GM}}} - 1}\]

but no choice of \( C \) (except \( C = -\infty \)) does this. But note that if \( 1 = 8GM \)

then \( p(R) = 0 \) for any \( C \).

We have found the odd result that in 2+1D, hydrostatic equilibrium of a circularly symmetric source requires \( 8GM = 1 \). This result can be shown to hold for any perfect fluid with a polytropic equation of state, i.e., \( p = k \rho^{1 + \frac{1}{n}} \), which encompasses a large array of sources.
2. Once across the event horizon of a Schwarzschild black hole, what is the longest proper time an observer can spend before reaching the singularity?

It would be useful to determine the rate \( \frac{dr}{d\tau} \) before proceeding.

\[
\frac{ds^2}{dt} = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\Omega^2 = -dt^2
\]

Then:

\[
(1 - \frac{2M}{r})\left(\frac{dt}{d\tau}\right)^2 = (1 - \frac{2M}{r})^{-1}\left(\frac{dr}{d\tau}\right)^2 = 1
\]

\[
\frac{dr}{d\tau} = \pm \sqrt{(\frac{2M}{r} - 1) + \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{dt}{d\tau}\right)^2}
\]

This term is \( > 0 \) so the smallest rate occurs when \( t = 0 \). The \(-1\) root is for decreasing \( r \)!

Or:

\[
\frac{d\tau}{dr} = \frac{-\frac{dr}{d\tau}}{\sqrt{(\frac{2M}{r} - 1)}} = \sqrt{\frac{2M}{r} - 1} = \int_{16\eta}^{\frac{16\eta}{r} - 1} \left[ \frac{1}{\sqrt{1 + \frac{2M}{r} - 1}} \right] = \frac{6M\eta}{\pi}\left(6M\eta - \frac{1}{2}\right) - 6M\eta\frac{1}{\pi}
\]

\( \approx 6M\pi \)
3. Consider the spacetime specified by the line element

\[ ds^2 = -(1 - \frac{GM}{r})^2 dt^2 + (1 - \frac{GM}{r})^{-2} dr^2 + r^2 d\Omega^2 \]

Except for \( r = GM \), the coordinate \( t \) is always timelike and the coordinate \( r \) is always spacelike.

a) Find a transformation to Eddington-Finkelstein-like coordinates \((v, r, \theta, \phi)\) such that \( g_{tt} = 0 \) and show that the geometry is not singular at \( r = GM \).

b) Sketch a plot analogous to our picture in class (EF for Schwarzschild) of the light cones in this geometry.

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a) We want a new "time" coordinate \( U \) s.t. \( g_{rr} = 0 \).

Assuming a form \( t = v + g(r) \Rightarrow dt = dv + \frac{dg}{dr} dr \)

Inserting this into the metric:

\[ ds^2 = -(1 - \frac{6H}{r})^2 (dv + \frac{dg}{dr} dr)^2 + (1 - \frac{6H}{r})^{-2} dr^2 + r^2 d\Omega^2 \]

Expanding and collecting all \( dr^2 \) coefficients we find:

\[ g_{rr} = -(1 - \frac{6H}{r})^2 (\frac{dg}{dr})^2 + (1 - \frac{6H}{r})^{-2} = 0 \Rightarrow \frac{dg}{dr} = \pm (1 - \frac{6H}{r})^{-\frac{1}{2}} \]

I chose the negative root (you can use either) and feeding to Mathematica yields: \( g(r) = -r - 26H \ln |r - 6H| + \frac{6H^3}{r - 6H} \)

\[ t = v - \frac{6H}{(r - 6H)} \]

\[ dt = dv - dr - \frac{6H}{(r - 6H)} \frac{dr}{(r - 6H)} \frac{G^{1/2}}{(r - 6H)^{1/2}} \]

\[ = dv - (1 + \frac{6H}{r - 6H} + \frac{G^{1/2}}{(r - 6H)^{1/2}} dr \]

\[ = dv - (\frac{1}{r - 6H} + \frac{6H}{r - 6H} + \frac{G^{1/2}}{(r - 6H)^{1/2}} (r - 6H)) \frac{dr}{(r - 6H)^{1/2}} \]

\[ = dv - \frac{r^{1/2} dr}{(r - 6H)^{1/2}} = dv - (1 - \frac{6H}{r})^{-1/2} dr \]

\[ ds^2 = - (1 - \frac{6H}{r})^2 dv^2 + (1 - \frac{6H}{r})^{-1} dv dr - \frac{1}{(1 - \frac{6H}{r})^{1/2}} dv^2 \]

\[ ds^2 = -(1 - \frac{6H}{r})^2 dv^2 + dv dr + r^2 d\Omega^2 \] which is non-singular at \( r = 6H \).
b) For radial null geodesics ($ds^2 = 0$, $u \partial_u = 0$):

\[ 0 = -\left(1 - \frac{\mathcal{M}}{r}\right)^2 du^2 + 2 \, du \, dr \]

Solutions include:

i) $du = 0 \Rightarrow dt = -(1 - \frac{\mathcal{M}}{r})^{-2} dr$

\[ \frac{dr}{dt} = -(1 - \frac{\mathcal{M}}{r})^2 \leq 0 \quad \text{for all } r \quad (\text{ingoing}) \]

ii) $\frac{dr}{du} = \frac{1}{2} (1 - \frac{\mathcal{M}}{r})^{-2} > 0 \quad \text{for all } r \quad (\text{outgoing})$ Different than the Schwarzschild case!!

iii) $dr = 0 \quad \text{w/ } r = \mathcal{M} \quad (\text{fixed at } r = \mathcal{M})$