General Relativity HW2 Problems

1. Inspired by our example of the group of 2D rotations that carry the corners of a square into corners, consider the set of 2D rotations that carry the corners of an equilateral triangle into themselves. Develop a three-dimensional faithful representation of this group and list both the "vectors" that correspond to the states, as well as the matrix transformations between them.

2. Show explicitly that the transformation matrix $\Lambda=\begin{pmatrix} \gamma & -\gamma\frac{v}{c}cos\phi & -\gamma\frac{v}{c}sin\phi & 0 \\ -\gamma\frac{v}{c} & \gamma cos\phi & \gamma sin\phi & 0 \\ 0 & -sin\phi & cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ where $\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ satisfies $\Lambda^T\eta\Lambda=\eta$. Describe what this transformation does in words.

Now that you have some familiarity with the Lorentz transformations you can use these to derive some classic results in special relativity. The following results can be derived using a simple boost transformation along the *x* axis. I know these are derived in many different texts, but try to do the following problems using what I give you in the problem itself and the explicit form of the boost transformation. My aim is to have you realize that from the perspective of 4D Lorentz transformations most of Special Relativity is quite straightforward even if counter-intuitive from the 3D perspective.

3. "Time-Dilation" A frame S' is moving with respect to frame S with a speed *v* along +*x*. Consider two events which in S have zero spatial separation, i.e. $\Delta x = \Delta y = \Delta z = 0$, but some nonzero Δt. These could be two ticks of a clock which is at rest in S, so we can call this Δt_{rest}. Note that in S' the clock will be moving along -*x* with speed *v*.
   Determine the time interval between the two events as measured in S', i.e. $\Delta t'$.
4. (*Challenge problem, i.e. will not be covered on quiz*) "Length-Contraction" A frame S' is moving with respect to frame S with a speed *v* along +*x*. Consider an object along *x*' which is at rest with respect to S', e.g. $|\Delta x'|=L_{rest}$. From the perspective of frame S, the object is moving along the *x* axis with a speed *v* and we an observer in S can make a measurement of its length by recording the time that the one end passes the origin and then the time when the other end passes the origin. This will yield a value Δt, i.e. a time interval in S. Using Δt and *v* an observer in S would calculate that $L = v\Delta t$. Thus the two events in S would have the coordinate separations $\Delta t = L/v$, $\Delta x=\Delta y=\Delta z=0$. Use this to determine the corresponding length in the frame S', i.e. $L = \Delta x'$.
5. Three events **A,B,C** are seen by an observe O to occur in the order **ABC**. Another Observer O' sees the same three events occur in the order **CBA**. Is it possible that a third observer O'' could see the events in the order **ACB**? Support your conclusions by drawing a spacetime diagram.
6. On a *ct*-*x* spacetime diagram, draw four events **A**,**B**, **C** and **D** such that **A** can cause **B** and **C**, **B** can cause **D** but not **C**, and **C** cannot cause **D**. Is such a situation possible in Galilean Relativity?
7. Prove that in special relativity $(\Lambda^{0\prime}{}_{0})^2 \geq 1$. **Hint:** Consider the equation that defines the Λ's in terms of the metric η.

8. Consider objects N_{ij} and M^{ij} in 2D with components:

$$N_{11} = a, N_{12} = b, N_{21} = c, N_{22} = d$$
$$M^{11} = e, M^{12} = f, M^{21} = g, M^{22} = h$$

Evaluate the following using index notation:

a) $N_{ij}M^{ki}$

b) $N_{ij}M^{kj}$

c) $N_{ij}M^{ji}$

d) $N_{ij}M^{ij}$

For each of the above, rewrite and evaluate using matrix operations when possible.