General Relativity HW3 Problems

1. The notion of the tangent space and cotangent space at a point are connected through the metric on the space. This pair is one example of dual spaces, but there are many others. Here is another example of a dual space: Consider the lattice of points in $\mathbb{R}^2$ which are at even integer coordinate positions, i.e. $(x, y) = (2n_x, 2n_y)$ where n_x, n_y are integers. We will call this the even lattice E^2. Now consider a dual lattice D^2 which is composed of points such that the Euclidean inner product of any lattice vector in D^2 with any lattice vector in E^2 always gives an integer. Identify the full set of points in this dual lattice. Note any differences between this dual pair and the tangent and cotangent spaces we have encountered.

2. For a 3D space in a particular set of coordinates the metric takes the form $g_{\mu\nu} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$

   Note that we do not use η for the metric unless we are in Minkowski space. More generally we will denote the metric by *g*.

   a) For a vector in this space with components (1,1,1) determine the components of the corresponding dual vector.
   b) For a dual vector with components (1,1,1) determine the components of the corresponding vector.
   c) Determine the "dot product" between the vector given in (a) with the dual vector given in (b).
   d) Determine the "dot product" between the vector given in (a) and its corresponding dual vector.
   e) Determine the "dot product" between the dual vector given in (b) and its corresponding vector.

3. Imagine we have a tensor $X^{\mu\upsilon}$ and a vector V^{μ} with components $X^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}$, $V^{\mu} = (-1, \quad 2, \quad 0, \quad -2)$. In this problem you may assume the Minkowski metric diag(-1,1,1,1). Find the components of:

   a) $X^{\mu}{}_{\nu}$
   b) $X_{\mu}{}^{\nu}$
   c) $X^{(\mu\nu)}$
   d) $X_{[\mu\nu]}$
   e) $X^{\lambda}{}_{\lambda}$
   f) $V^{\mu}V_{\mu}$

   Note: $(\mu\nu)$ means construct $\frac{1}{2}\mu\nu + \frac{1}{2}\nu\mu$ while $[\mu\nu]$ means construct $\frac{1}{2}\mu\nu - \frac{1}{2}\nu\mu$.

4. Starting from $\partial_{[\mu}F_{\nu\lambda]} = 0$ derive the corresponding Maxwell's equations in terms of 3-component vector quantities $\vec{E}$ and $\vec{B}$. **Hint:** To antisymmetrize three indices you should construct $[\mu\nu\lambda] = \frac{1}{3!}(\mu\nu\lambda + \lambda\mu\nu + \nu\lambda\mu - \nu\mu\lambda - \lambda\nu\mu - \mu\lambda\nu)$.