General Relativity HW 4 Quiz   Name KEY

You know the drill!

1. (*10pts*) Consider a particle moving along a path parameterized by λ and described by $x^\mu(\lambda) = (\lambda, (\lambda-1)^2, 2, -\lambda)$. Also consider a function over space-time which is given by $f(t,x,y,z) = t^2 + xy - z^2$. Find the value of λ for which $\frac{df}{d\lambda} = 0$.

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial x^\mu}\frac{\partial x^\mu}{\partial \lambda}$$

$$\frac{\partial f}{\partial x^\mu} = (2t, y, x, -2z)$$

$$\frac{\partial x^\mu}{\partial \lambda} = (1, 2\lambda-2, 0, -1)$$ using $t=\lambda, y=2, z=-\lambda$

$$\frac{df}{d\lambda} = 2t + y(2\lambda-2) + 2z = \overbrace{2\lambda + 2(2\lambda-2) - 2\lambda}$$

$$\frac{df}{d\lambda} = 4\lambda - 4 = 0 \Rightarrow \lambda = 1$$

Turn over for second question.

2. (*10pts*) Consider the energy-momentum tensor $T^{\mu\nu}$ of a perfect fluid with equation of state $p = \frac{1}{3}\rho$. Find an explicit expression for $T^{\mu\nu}$ in a frame boosted along the x-axis with a speed v with respect to the overall rest frame of the fluid. Express your answer in terms of ρ and v. Your answer may include γ-factors as well.

Method 1: $$T^{\mu\nu}_{rest} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} = \begin{pmatrix} \rho & & & \\ & \frac{1}{3}\rho & & \\ & & \frac{1}{3}\rho & \\ & & & \frac{1}{3}\rho \end{pmatrix}$$

$$T^{\mu'\nu'} = \Lambda T_{rest} \Lambda^T = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho & & & \\ & \frac{1}{3}\rho & & \\ & & \frac{1}{3}\rho & \\ & & & \frac{1}{3}\rho \end{pmatrix} \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma\rho & -\frac{1}{3}v\gamma\rho & 0 & 0 \\ -v\gamma\rho & \frac{1}{3}\gamma\rho & 0 & 0 \\ 0 & 0 & \frac{1}{3}\rho & 0 \\ 0 & 0 & 0 & \frac{1}{3}\rho \end{pmatrix} \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^2\rho + \frac{1}{3}v^2\gamma^2\rho & -v\gamma^2\rho - \frac{1}{3}v\gamma^2\rho & 0 & 0 \\ -v\gamma^2\rho - \frac{1}{3}v\gamma^2\rho & v^2\gamma^2\rho + \frac{1}{3}\gamma^2\rho & 0 & 0 \\ 0 & 0 & \frac{1}{3}\rho & 0 \\ 0 & 0 & 0 & \frac{1}{3}\rho \end{pmatrix}$$

Method 2: $$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p\eta^{\mu\nu} = \frac{4}{3}\rho u^\mu u^\nu + \frac{1}{3}\rho\eta^{\mu\nu}$$

Using $u^\mu = (\gamma, -v\gamma, 0, 0)$ this becomes

$$T^{\mu\nu} = \frac{4}{3}\rho \begin{pmatrix} \gamma^2 & -v\gamma^2 & 0 & 0 \\ -v\gamma^2 & v^2\gamma^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{3}\rho & & & \\ & \frac{1}{3}\rho & & \\ & & \frac{1}{3}\rho & \\ & & & \frac{1}{3}\rho \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{3}\gamma^2\rho - \frac{1}{3}\rho & -\frac{4}{3}v\gamma^2\rho & 0 & 0 \\ -\frac{4}{3}v\gamma^2\rho & \frac{4}{3}v^2\gamma^2\rho + \frac{1}{3}\rho & 0 & 0 \\ 0 & 0 & \frac{1}{3}\rho & 0 \\ 0 & 0 & 0 & \frac{1}{3}\rho \end{pmatrix}$$

Comparing w/ above, note $v^2 = 1 - \frac{1}{\gamma^2}$ so

$\gamma^2\rho + \frac{1}{3}v^2\gamma^2\rho = \gamma^2\rho + \frac{1}{3}\gamma^2\rho - \frac{1}{3}\rho$

$= \frac{4}{3}\gamma^2\rho - \frac{1}{3}\rho$ ✓

and

$v^2\gamma^2\rho + \frac{1}{3}\gamma^2\rho = \gamma^2\rho + \frac{1}{3}\rho + \frac{1}{3}\gamma^2\rho$

$= \frac{4}{3}\gamma^2\rho + \frac{1}{3}\rho$ ✓