General Relativity HW5 Problems

1. Imagine a particle following a path through spacetime given by \( x^\mu (\tau) = \left( \tau^2 + \tau, \tau^2, \frac{4}{3} \tau^2, -10 \right) \).

   a) Compute the four-velocity of the particle as it passes through the point \( x^\mu = (20, 16, \frac{32}{3}, -10) \).

   b) For the function \( f(t, x, y, z) = -t^2 + x^2 + y^2 - yz \), calculate the rate of change of this function along the path from part (a), i.e. \( \frac{df}{d\tau} \), at the point \( x^\mu = (20, 16, \frac{32}{3}, -10) \).

      Hint: You will need to break up the derivative into two terms using \( \frac{dx^\mu}{d\tau} \) in various places so that can use your result for the four-velocity.

2. The energy-momentum tensor of a perfect fluid in its rest frame is given by \( T^\mu_\nu = \text{diag}(\rho, p, p, p) \).

   Find a matrix expression for the energy-momentum tensor seen by an observer moving with a speed \( v \) along the \( e_{(1)} + e_{(2)} \) direction. Do this in two ways:

   a) Use Lorentz transformations to explicitly transform \( T^\mu_\nu \).

   b) Use the expression (valid in any frame) \( T^\mu_\nu = (\rho + p) U^\mu U^\nu + p \eta^\mu_\nu \).

3. Most energy-momentum tensors naturally specify a preferred inertial frame for which the overall system is at rest. For the perfect fluid case, this is typically the frame for which the matrix realization of the tensor is diagonal. Consider the case of vacuum energy with an equation of state \( p = -\rho \). Treating this as perfect fluid, what can you say about the preferred rest frame of the vacuum energy system?