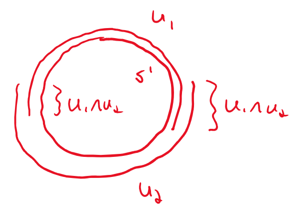
General Relativity HW5 Problems

1. Recall my argument from class in which I said that covering a manifold by two charts each of which entirely covers the space means that the transition functions must be well defined throughout the space. Since a circle cannot be covered entirely by a single chart, how would we reach this conclusion for a circle.

So in class we constructed an atlas for S^1 using

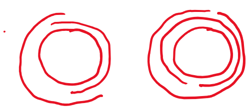


$\Rightarrow$ The overlaps are the regions on $U_1 \cap U_2$ and this is where transition functions are defined.

In order to extend the definition and properties (C^P) of the transition functions to the entire manifold S^1, recall that no single atlas defines it, but rather the maximal atlas, which includes all atlases.

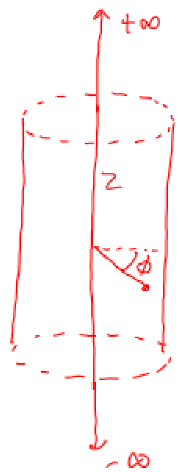
So we can simply use other atlases to cover the remaining parts of S^1!

E.g.



etc.

2. Given that a circle requires at least two charts to form an atlas, it might be surprising that the surface of an infinite cylinder can be covered with an atlas consisting of only one chart. Construct such an atlas for the cylinder. Remember, this is now a 2D space.

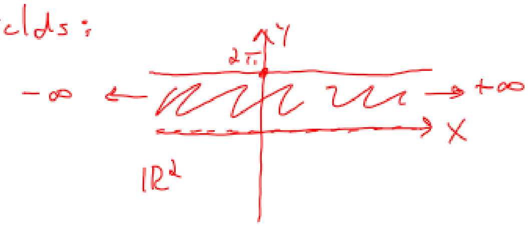


Label points in the space w/ $z \in (-\infty, \infty)$

$\phi \in (0, 2\pi]$

You might be tempted to try the map ϕ: $x = z$, $y = \phi$ into $\mathbb{R}^2$

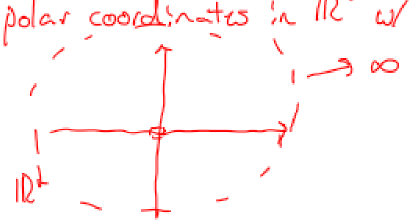
but this yields:



But this is not open since it includes the boundary line at $y = 2\pi$.
Remember, for a good chart the image in $\mathbb{R}^n$ must be open!

But we can instead use polar coordinates in $\mathbb{R}^2$ w/ ϕ: $\underbrace{r = e^{z}, \ \theta = \phi}$

Then our image is



ote $r \in (0, \infty)$

$\theta \in (0, 2\pi]$

Which is open!

For the next two questions, recall that the interval ds^2 for the coordinate displacements $dx^\mu = (dx^1, dx^2, \ldots)$ can be obtained from the metric $g_{\mu\nu}$ by $ds^2 = dx^\mu g_{\mu\nu} dx^\nu$.

3. Consider $\mathbb{R}^3$ as a manifold with the flat Euclidean metric, and coordinates $\{x, y, x\}$. Introduce spherical polar coordinates $\{r, \theta, \phi\}$ related to $\{x, y, x\}$ by

$$x = r\, sin\theta\, cos\phi$$
$$y = r\, sin\theta\, sin\phi$$
$$z = r\, cos\theta$$

so that the metric takes the form

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2.$$

a) If a particle moves along the parameterized curve given by

$$x(\lambda) = cos\lambda \quad y(\lambda) = \sin\lambda \quad z(\lambda) = \lambda$$

express the path of the curve in the $\{r, \theta, \phi\}$ coordinate system.

b) Calculate the components of the tangent vector to the curve in both the Cartesian and spherical polar coordinate systems.

$\mathbb{R}^3$

a. $x = r\sin\theta\cos\phi$ $\qquad x(\lambda) = \cos\lambda$

$y = r\sin\theta\sin\phi$ $\qquad y(\lambda) = \sin\lambda$

$z = r\cos\theta$ $\qquad z(\lambda) = \lambda$

$\Downarrow$ $\qquad \Downarrow$

$r = \sqrt{x^2+y^2+z^2}$ $\qquad r(\lambda) = \sqrt{1+\lambda^2}$

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$ $\qquad \phi(\lambda) = \tan^{-1}(\tan\lambda) = \lambda$

$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$ $\qquad \theta(\lambda) = \tan^{-1}\left(\frac{1}{\lambda}\right)$

b. $\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}$

components, so

in (r, ϕ, λ)

$\frac{dr}{d\lambda} = (1+\lambda^2)^{-1/2}$

$\frac{d\phi}{d\lambda} = 1$

$\frac{d\theta}{d\lambda} = \frac{1}{1+\frac{1}{\lambda^2}}\left(-\frac{1}{\lambda^2}\right) = \frac{-1}{1+\lambda^2}$

in (x, y, z)

$\frac{dx}{d\lambda} = -\sin\lambda$

$\frac{dy}{d\lambda} = \cos\lambda$

$\frac{dz}{d\lambda} = 1$

4. Prolate spheroidal coordinates are related to the usual Cartesian coordinates $\{x, y, x\}$ of Euclidean three-space by

$$x = \sinh\chi \sin\theta \cos\phi$$
$$y = \sinh\chi \sin\theta \sin\phi$$
$$z = \cosh\chi \cos\theta$$

Restrict your attention to the $y = 0$ plane and answer the following:

a) What is the coordinate transformation matrix $\frac{\partial x^{\mu}}{\partial x^{\mu'}}$ relating $\{x, z\}$ to $\{\chi, \theta\}$.

b) What does the invariant interval ds^2 look like in prolate spheroidal coordinates?

In $\mathbb{R}^3$ we have: $\left.\begin{array}{l} x = \sinh\chi \sin\theta \cos\phi \\ y = \sinh\chi \sin\theta \sin\phi \\ z = \cosh\chi \cos\theta \end{array}\right\}$ for $y=0$, $\phi = 0$ $\Rightarrow \begin{cases} x = \sinh\chi \sin\theta \\ z = \cosh\chi \cos\theta \end{cases}$

a)

To get $\frac{\partial x^{\mu}}{\partial x^{\nu'}}$ (μ: row, ν': column) we use $\begin{pmatrix} \frac{\partial x}{\partial \chi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial z}{\partial \chi} & \frac{\partial z}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cosh\chi \sin\theta & \sinh\chi \cos\theta \\ \sinh\chi \cos\theta & -\cosh\chi \sin\theta \end{pmatrix}$

b) 2 ways to do this: $x = \sinh\chi \sin\theta \Rightarrow dx = \cosh\chi \sin\theta \, d\chi + \sinh\chi \cos\theta \, d\theta$

$z = \cosh\chi \cos\theta \Rightarrow dz = \sinh\chi \cos\theta \, d\chi - \cosh\chi \sin\theta \, d\theta$

Then: $ds^2 = dx^2 + dz^2$

$= (\cosh\chi \sin\theta \, d\chi + \sinh\chi \cos\theta \, d\theta)^2 + (\sinh\chi \cos\theta \, d\chi - \cosh\chi \sin\theta \, d\theta)^2$

$= \cosh^2\chi \sin^2\theta \, d\chi^2 + \sinh^2\chi \cos^2\theta \, d\theta^2 + 2\cosh\chi \sin\theta \sinh\chi \cos\theta \, d\chi \, d\theta$

$+ \sinh^2\chi \cos^2\theta \, d\chi^2 + \cosh^2\chi \sin^2\theta \, d\theta^2 - 2\cosh\chi \sin\theta \sinh\chi \cos\theta \, d\chi \, d\theta$

$= (\cosh^2\chi \sin^2\theta + \sinh^2\chi \cos^2\theta) \, d\chi^2 + (\cosh^2\chi \sin^2\theta + \sinh^2\chi \cos^2\theta) \, d\theta^2$

$= (\cosh^2\chi \sin^2\theta + \sinh^2\chi \cos^2\theta)(d\chi^2 + d\theta^2)$

Or you can say: $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow g_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\mu\nu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} g_{\mu\nu} \frac{\partial x^{\nu}}{\partial x^{\nu'}}$

Then: $g_{\mu'\nu'} = \begin{pmatrix} \cosh\chi \sin\theta & \sinh\chi \cos\theta \\ \sinh\chi \cos\theta & -\cosh\chi \sin\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\chi \sin\theta & \sinh\chi \cos\theta \\ \sinh\chi \cos\theta & -\cosh\chi \sin\theta \end{pmatrix}$

$= \begin{pmatrix} \cosh^2\chi \sin^2\theta + \sinh^2\chi \cos^2\theta & \cancel{\cosh\chi \sinh\chi \sin\theta \cos\theta - \cosh\chi \sinh\chi \sin\theta \cos\theta} \rightarrow 0 \\ \cancel{\sinh\chi \cosh\chi \cos\theta \sin\theta - \sinh\chi \cosh\chi \cos\theta \sin\theta} \rightarrow 0 & \sinh^2\chi \cos^2\theta + \cosh^2\chi \sin^2\theta \end{pmatrix}$

$= (\cosh^2\chi \sin^2\theta + \sinh^2\chi \cos^2\theta) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then: $ds^2 = (d\chi \;\; d\theta) \begin{pmatrix} g_{\mu'\nu'} \end{pmatrix} \begin{pmatrix} d\chi \\ d\theta \end{pmatrix} = (\cosh^2\chi \sin^2\theta + \sinh^2\chi \cos^2\theta)(d\chi^2 + d\theta^2)$

Same!