1. Imagine a particle following a path through spacetime given by \( x^\mu(\tau) = \left( \tau^2 + \tau, \tau^2, \frac{4}{3} \tau^2, -10 \right) \).

   a) Compute the four-velocity of the particle as it passes through the point \( x^\mu = (20, 16, \frac{32}{3}, -10) \).

   First note that the point under consideration corresponds to \( \tau = 4 \).

   Then we want \( U^\mu(\tau = 4) = \frac{dx^\mu}{d\tau} \bigg|_{\tau=4} \).

   \[
   U^\mu(\tau) = \left( 2\tau + 1, 2\tau, 2\tau^2, 0 \right)
   \]

   Thus:

   \[
   U^\mu(\tau = 4) = \left( 9, 8, 4, 0 \right)
   \]

   b) For the function \( f(t, x, y, z) = -t^2 + x^2 + y^2 - yz \), calculate the rate of change of this function along the path, i.e. \( \frac{\partial f}{\partial \tau} \), at the point \( x^\mu = (20, 16, \frac{32}{3}, -10) \).

   Hint: You will need to break up the directional derivative into two terms using \( \frac{\partial x^\mu}{\partial \tau} \) in various places so that you can use your result for the four-velocity.

   To evaluate \( \frac{\partial f}{\partial \tau} \), consider \( \frac{\partial f}{\partial \tau} = \frac{\partial f}{\partial x^\mu} \frac{\partial x^\mu}{\partial \tau} = \frac{\partial f}{\partial x^\mu} U^\mu \).

   First:

   \[
   \frac{\partial f}{\partial x^\mu} = \left( \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)
   \]

   \[
   = (-2t, 2x, 2y - z, -y)
   \]

   Then:

   \[
   \frac{\partial f}{\partial x^\mu} U^\mu = \left( -2t, 2x, 2y - z, -y \right) \left( \frac{2\tau + 1}{2\tau^2}, \frac{2\tau}{2\tau^2}, \frac{2\tau^2}{2\tau^2}, 0 \right)
   \]

   \[
   = -2t(2\tau + 1) + 4x \tau + 4y \tau^2 - 2z \tau^2
   \]

   Then at \( \tau = 4 \) or \( t = 20 \), \( x = 16 \), \( y = \frac{32}{3} \), \( z = -10 \) we have

   \[
   \frac{\partial f}{\partial \tau} \bigg|_{\tau=4} = -400 + 256 + 85.33 + 40 = 21.33
   \]
2. The energy-momentum tensor of a perfect fluid in its rest frame is given by $T^{\mu\nu}=\text{diag}(\rho,p,p,p)$. Find a matrix expression for the energy-momentum tensor seen by an observer moving with a speed $v$ along the $e_{(1)}+e_{(2)}$ direction. Do this in two ways:

a) Use Lorentz transformations to explicitly transform $T^{\mu\nu}$.

We need the matrix form of a boost along $e_{(1)}+e_{(2)}$. To get this, we can first rotate $\mathbb{R}^4$ by $\frac{\pi}{2}$ to $\mathbb{R}^3$, then boost along $x'$ (which we know the form for), then rotate back to $\mathbb{R}^4$ to see what the result looks like in the original spatial coordinates.

Before we do that, let's consider how $T^{\mu\nu}$ transforms in general.

$$T^{\mu\nu} \rightarrow T'^{\mu\nu} = \Lambda^\mu_{\lambda'} \Lambda^\nu_{\mu'} T^{\lambda\mu}, \quad \Lambda^\mu_{\lambda'} \Lambda^\nu_{\mu'} = \Lambda^T \Lambda^T \Lambda \Lambda^T$$

Doing our transformation in 3 steps, we find:

$$T^{\mu\nu} \rightarrow T'^{\mu\nu} = \Lambda^\mu_{\lambda'} \Lambda^\nu_{\mu'} T^{\lambda\mu} \overset{\text{Lorentz}}{\rightarrow} \lambda_{R_{\nu}(\phi,\psi)} \lambda_{R_{\lambda}(\phi,\psi)} \lambda^T_{R_{\lambda}(\phi,\psi)} \lambda^T_{R_{\nu}(\phi,\psi)}$$

Since $T_{\mu\nu} = \left(\begin{array}{cccc} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{array} \right)$ which is isotropic in $e_{(1)}$, so we know that $R_{\nu}(\phi,\psi)$ will not change anything! You don't have to realize this. Just multiply it out!

Then using $\lambda_{B_{(1,2)}} = \left(\begin{array}{cccc} \gamma & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$ and $\lambda_{R_{\nu}(\phi,\psi)} = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$

We have:

$$T^{\mu\nu} = \left(\begin{array}{cccc} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{array} \right) \rightarrow T'^{\mu\nu} = \left(\begin{array}{cccc} \rho \gamma & \rho & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{array} \right)$$

For comparison to the next part, let's rewrite a few terms using $v^2 = \frac{1}{\gamma^2}$ or $u^2 = \frac{-1}{\gamma^2}$:

$$T_{\eta\nu}' = \rho \gamma u v + p u^2 = \rho u v + p u^2$$

b) Use the expression (valid in any frame) $T^{\mu\nu} = (\rho+p) U^\mu U^\nu + p \eta^{\mu\nu}$.

Using the observer moving along $e_{(1)}+e_{(2)}$ at speed $v$ see the fluid moving along $-e_{(1)}-e_{(3)}$ w/ $u$:

$$U^\mu = \left(\begin{array}{cccc} \gamma & 0 & 0 & 0 \\ \frac{1}{\gamma} (v_1 - \frac{v_2}{2}) & \frac{1}{\gamma} (v_1 + \frac{v_2}{2}) & \frac{1}{\gamma} v_3 & 0 \\ \frac{1}{\gamma} (v_1 + \frac{v_2}{2}) & \frac{1}{\gamma} (v_1 - \frac{v_2}{2}) & \frac{1}{\gamma} v_3 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

Then: $T^{\mu\nu} = (\rho+p) U^\mu U^\nu + p \eta^{\mu\nu}$
3. Most energy-momentum tensors naturally specify a preferred inertial frame for which the overall system is at rest. For the perfect fluid case, this is typically the frame for which the matrix realization of the tensor is diagonal. Consider the case of vacuum energy with an equation of state $p = -\rho$. Treating this as perfect fluid, what can you say about the preferred rest frame of the vacuum energy system?

For vacuum we have $T^{\mu\nu}_{\text{vac}} = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix} = \begin{pmatrix} -\rho & 0 \\ 0 & \rho \end{pmatrix} = \rho \pi^{\mu\nu}$

But we know that SR is based on transformations between (inertial) frames which leave $\eta_{\mu\nu}$ (hence $\pi^{\mu\nu}$) invariant, so we find that $T^{\mu\nu}_{\text{vac}}$ is also invariant.

This means that unlike a normal perfect fluid where it makes something to be at rest w.r.t. the fluid (and hence $T^{\mu\nu}$ transforms to another form when boosted), it doesn’t make any sense to be at rest w.r.t. the vacuum. All observers are equal w.r.t. to the vacuum, so they should (and do!) see the same form for $T^{\mu\nu}_{\text{vac}}$. 