General Relativity HW 6 Quiz

Name KEY

You know the drill!

1. For a 2-sphere with coordinates (θ, ϕ), write down the equations for parallel transport of a vector along a line of constant **longitude.** Then parallel transport the vector with components $V^\mu = (1,0)$ once around the line and write down the result. You may use any results from your homework without deriving them again.

For S^2 in (θ, ϕ) we have: $\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta$, $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$

For //-transport of V^μ we require: $\frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0$

For a line of constant longitude: $x^\mu(\lambda) = (\lambda, \phi_0) \Rightarrow \frac{d\theta}{d\lambda} = 1,\ \frac{d\phi}{d\lambda} = 0$

Then:

$$\theta:\ \frac{dV^\theta}{d\lambda} + \Gamma^\theta_{\phi\phi}\cancelto{0}{\frac{d\phi}{d\lambda}}V^\phi = \frac{dV^\theta}{d\lambda} = 0 \quad\Rightarrow\quad V^\theta = \text{constant}$$

$$\phi:\ \frac{dV^\phi}{d\lambda} + \Gamma^\phi_{\theta\phi}\frac{d\theta}{d\lambda}V^\phi + \Gamma^\phi_{\phi\theta}\cancelto{0}{\frac{d\phi}{d\lambda}}V^\theta = \frac{dV^\phi}{d\lambda} + \cot\theta\, V^\phi = 0$$

$$\frac{dV^\phi}{d\lambda} = -\cot\theta\, V^\phi \Rightarrow V^\phi = Ae^{-\cot\theta\,\lambda} \Rightarrow V^\phi(\lambda=0) = 0 \Rightarrow A = 0$$

So $V^\mu_{//} = (1,0)$

Alternatively:

$V^\mu = (1,0)$ is tangent to the curve $x^\mu(\lambda) = (\lambda, \phi_0)$ <u>and</u> $x^\mu(\lambda)$ is a geodesic, so the tangent vector should not change as it is //-transported along $x^\mu(\lambda)$.

Turn over for second question.

2. Consider the upper-half plane model of the hyperbolic plane $H = \{(x, y) \in \mathbb{R}^2 | y > 0\}$ with line element $ds^2 = \frac{dx^2+dy^2}{y^2}$. Find the form of the divergence operator on a vector function $V^\mu(x, y)$ in the coordinate basis.

Using $x^1 = x$, $x^2 = y$

$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda = \partial_1 V^1 + \partial_2 V^2 + \Gamma^1_{11} V^1 + \Gamma^2_{21} V^1 + \Gamma^1_{12} V^2 + \Gamma^2_{22} V^2$

Note: $g_{\mu\nu} = \begin{pmatrix} y^{-2} & 0 \\ 0 & y^{-2} \end{pmatrix}$ so we can use the results of problem 3.

$\Gamma^1_{11} = \partial_1 (\ln \sqrt{|g_{11}|}) = \partial_x (\ln \sqrt{|y^{-2}|}) = 0$

$\Gamma^2_{21} = \partial_1 (\ln \sqrt{|g_{22}|}) = \partial_x (\ln \sqrt{|y^{-2}|}) = 0$

$\Gamma^1_{12} = \partial_2 (\ln \sqrt{|g_{11}|}) = \partial_y (\ln (\frac{1}{y})) = {-1/y^2}/{1/y} = -\frac{1}{y}$

$\Gamma^2_{22} = \partial_2 (\ln \sqrt{|g_{22}|}) = \partial_y (\ln (\frac{1}{y})) = -\frac{1}{y}$

Then: $\boxed{\nabla_\mu V^\mu = \partial_x V^x + \partial_y V^y - \frac{2}{y} V^y}$