1.  (10pts) Prolate spheroidal coordinates are related to the usual Cartesian coordinates \( x, y, z \) of Euclidean three-space by

\[
\begin{align*}
x &= \sinh \chi \sin \theta \cos \phi \\
y &= \sinh \chi \sin \theta \sin \phi \\
z &= \cosh \chi \cos \theta
\end{align*}
\]

What does the invariant interval \( ds^2 \) look like in prolate spheroidal coordinates when \( \theta = \frac{\pi}{2} \)?

When \( \theta = \frac{\pi}{2} \) we have \( x = \sinh \chi \cos \phi, \quad y = \sinh \chi \sin \phi, \quad z = 0 \)

There are 2 ways to find \( ds^2 \):

a) \( dx = \frac{\partial x}{\partial \chi} \, d\chi + \frac{\partial x}{\partial \theta} \, d\theta + \frac{\partial x}{\partial \phi} \, d\phi = \cosh \chi \cos \theta \, d\chi \cos \phi - \sinh \chi \sin \phi \, d\phi \)

\( dy = \frac{\partial y}{\partial \chi} \, d\chi + \frac{\partial y}{\partial \theta} \, d\theta + \frac{\partial y}{\partial \phi} \, d\phi = \cosh \chi \sin \theta \, d\chi \sin \phi + \sinh \chi \cos \phi \, d\phi \)

Then \( ds^2 = dx^2 + dy^2 = (\cosh \chi \cos \theta \, d\chi \cos \phi - \sinh \chi \sin \phi \, d\phi)^2 + (\cosh \chi \sin \theta \, d\chi \sin \phi + \sinh \chi \cos \phi \, d\phi)^2 \)

\( ds^2 = \cosh \chi \, d\chi^2 + \sinh \chi \, d\phi^2 \)

b) \( g_{\mu \nu} \rightarrow g_{\mu' \nu'} = \frac{\partial x^\mu}{\partial x'^{\mu'}} \frac{\partial x^\nu}{\partial x'^{\nu'}} g_{\mu \nu} \) \quad \text{with} \quad \frac{\partial x^\mu}{\partial x'^{\mu'}} = \begin{pmatrix} \frac{\partial x^\chi}{\partial x'^{\chi}} & \frac{\partial x^\theta}{\partial x'^{\theta}} & \frac{\partial x^\phi}{\partial x'^{\phi}} \\ \frac{\partial x^\theta}{\partial x'^{\chi}} & \frac{\partial x^\theta}{\partial x'^{\theta}} & \frac{\partial x^\phi}{\partial x'^{\phi}} \\ \frac{\partial x^\phi}{\partial x'^{\chi}} & \frac{\partial x^\phi}{\partial x'^{\theta}} & \frac{\partial x^\phi}{\partial x'^{\phi}} \end{pmatrix} = \begin{pmatrix} \cosh \chi \cos \theta - \sinh \chi \sin \phi & 0 & 0 \\ 0 & \cosh \chi \sin \theta \sin \phi & \sinh \chi \cos \phi \\ 0 & \sinh \chi \sin \phi & \cosh \chi \cos \phi \end{pmatrix}

\( x^\mu = (x, \phi) \rightarrow x'^{\mu'} = (x', \gamma) \)

Then \( g_{\mu' \nu'} = \begin{pmatrix} \cosh \chi \cos \phi & 0 & 0 \\ 0 & \cosh \chi \cos \phi & 0 \\ 0 & 0 & \cosh \chi \cos \phi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cosh \chi \cos \phi & 0 & 0 \\ 0 & \cosh \chi \sin \phi \sin \phi & \sinh \chi \cos \phi \\ 0 & \sinh \chi \sin \phi & \cosh \chi \cos \phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sinh \chi \cos \phi + \cosh \chi \cos \phi & 0 \\ 0 & 0 & \sinh \chi \sin \phi \end{pmatrix} \)

Finally, \( ds^2 = (dx \, d\phi) \, g_{\mu \nu} \, (dx) = \cosh \chi \, dx^2 + \sinh \chi \, d\phi^2 \)
2. (10pts) Consider the open annulus which is the set of points in $\mathbb{R}^2$ such that $a < r < b$, when $\mathbb{R}^2$ is described in terms of polar coordinates $(r, \theta)$. Show that this space is a manifold that can be covered by a single chart. In your answer make sure you provide the explicit chart map.

The open annulus is $r \in (a, b), \theta \in [0, 2\pi]$ in $\mathbb{R}^2$ or the region outside of a circle of radius $a$, but inside a circle of radius $b$.

For a chart we only need a map from points in the annulus into $\mathbb{R}^2$, but the annulus itself is defined in $\mathbb{R}^2$ so the identity map will do!