General Relativity HW7 Problems

1. Find explicit expressions for all of the Killing vectors $K^\mu$ for Minkowski space $\mathrm{M}^4$. Be careful to recall that what appears in Killing's equation are the dual Killing vectors!

   For the following questions you may find it useful to use the Mathematica package G.R.E.A.T. I will be posting solutions as Mathematica notebooks using the package. You are free to use other resources if you like, but my familiarity is limited to Mathematica. You can download the package from the course website. Also, I have a sample notebook of how to use and read the output of that package.

2. The metric on a three-sphere in coordinates $x^\mu = (\psi, \theta, \phi)$ can be written as:
$$ds^2 = d\psi^2 + sin^2\psi(d\theta^2 + sin^2\theta d\phi^2)$$
   **Note**: This not a ball in $\mathbb{R}^3$, but rather a "surface" in $\mathbb{R}^4$ defined by $x^2 + y^2 + z^2 + w^2 = 1$.
   a. Calculate the Christoffel connection coefficients using whatever method you like.
   b. Calculate the Riemann tensor, Ricci tensor and Ricci scalar.
   c. Show that this space satisfies the expression $R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})$ where $n$ is the dimension of the space.
3. This is your first project dealing with a space that is not quite trivial and whose metric you will have to figure out on your own. It's time to think about the two-torus, i.e. what many of you might call a "doughnut". A general n-torus can be defined as an n-dimensional rectangle (a patch of Euclidean space $R^n$) with the opposite sides identified. So a 1D torus is a circle, a 2D torus is the usual "doughnut", and in 3D and higher it gets hard to visualize. By this definition the torus it is intrinsically flat since it starts with a patch of $R^n$ (which is flat) and simply adds nontrivial identifications (periodicity of the edges).

   However you may suspect that trying to build a two-torus that lives in 3D Euclidean space by this definition is a bit of a problem. Consider taking a piece of flat paper (a rectangular patch of Euclidean $R^2$ ) and forming a torus. We can easily roll the paper into a cylinder without "stretching" it. But to finish the torus we have to "roll" the cylinder so that the two circles on the ends meet. But you cannot do this without stretching or shrinking the paper, i.e. changing the distance between points, i.e. changing the metric! Essentially, the two-torus as we have defined it (by identifications and hence intrinsically flat) cannot live in 3D Euclidean space (though it can live in 4D or higher).

   To help make this clear, consider the 2D surface of a "doughnut" living in 3D. We won't call this a torus anymore. Construct a set of good coordinates for this doughnut surface and determine the metric in those coordinates. Hint: One way to do this is to use the embedding map into 3D, (x,y,z) = $((R_1 + R_2\sin v)\cos u, (R_1 + R_2\sin v)\sin u, R_2\cos v)$ where (u,v) are the two coordinates on the doughnut. Then using the line element in 3D Euclidean space for (x,y,z), you can convert this into a line element for (u,v). **Hint:** Just plug in the embedding map above. Finally calculate the

Riemann curvature tensor for your 2D metric, which will answer the question "Is the doughnut flat?" You will need the Mathematica G.R.E.A.T. package for this!

Lastly, to see that the flat two-torus can consistently live in 4D Euclidean space, consider the embedding map (x,y,z,w) = ($R_1$sin u, $R_1$cos u, $R_2$sin v, $R_2$cos v). Construct the 2D metric for this torus and then compute the Riemann curvature tensor from it.

4. Typically what we do with Einstein's equations is take a given distribution of sources, i.e. the energy-momentum tensor $T_{\mu\nu}$, and then solve for the resulting metric $g_{\mu\nu}$. However we can turn this around and ask, "given a particular geometry, i.e. a metric, what would be the required energy-momentum tensor to satisfy Einstein's equation?" This is a much easier task!
To see this consider the following metric:

$$ds^2 = -cosh^2(\psi)dt^2 + d\psi^2 + sinh^2(\psi)d\chi^2 + sinh^2(\psi)sin^2(\chi)d\delta^2 + d\eta^2$$
$$+sin^2(\eta)d\theta^2 + sin^2(\eta)sin^2(\theta)d\phi^2 + sin^2(\eta)sin^2(\theta)sin^2(\phi)d\beta^2$$

Calculate the energy-momentum tensor that would create this geometry. Also, calculate the trace of this energy-momentum tensor.