1. As a warm-up to the GR case, you will determine the transformation rule for the electromagnetic gauge field as a result of demanding covariance of the Dirac equation. Don’t worry if a lot of these words don’t make sense initially. In fact you will not really need to understand most of them to do the what I am asking.

   a. Consider the equation \( \gamma^\mu \partial_\mu \psi + \frac{mc}{\hbar} \psi = 0 \) where \( \psi(x^\mu) \) is a spinor field and \( \gamma^\mu \) are the constant Dirac matrices. The rest of the terms in this equation should be familiar. This equation is "covariant" under a transformation of the form \( \psi \rightarrow \psi' = e^{iq\varphi} \psi \) where \( q \) and \( \varphi \) are constants. What covariant means is that the entire left hand side of the equation transforms as \( \gamma^\mu \partial_\mu \psi + \frac{mc}{\hbar} \psi \rightarrow e^{iq\varphi}(\gamma^\mu \partial_\mu \psi + \frac{mc}{\hbar} \psi) \) which you can easily verify. Now what we want to do is make this equation covariant even when we allow \( \varphi \) to depend on position, i.e. \( \varphi(x^\mu) \). You can check that in its current form this equation is not covariant under this "local" transformation since \( \partial_\mu \psi \rightarrow \partial_\mu(e^{iq\varphi(x^\mu)} \psi) \) and the derivative will now act on both \( \varphi(x^\mu) \) and \( \psi(x^\mu) \). To fix this, we will make use of a new derivative of the form \( D_\mu \equiv \partial_\mu + iqA_\mu \) where \( A_\mu(x^\mu) \) will end up being the electromagnetic 4-vector potential (or gauge field). Your job is to figure out how the gauge field itself must transform in order for the equation with the new derivative to be covariant. That is, what does \( A'_\mu(x^\mu) \) look like in terms of \( A_\mu(x^\mu) \) and other quantities such that \( \gamma^\mu D_\mu \psi + \frac{mc}{\hbar} \psi \rightarrow e^{iq\varphi(x^\mu)}(\gamma^\mu D_\mu \psi + \frac{mc}{\hbar} \psi) \).

   b. Now, turning to GR, derive the transformation rule for the connection \( \Gamma^{\nu}_{\mu\lambda} \) from insisting that \( \partial_\mu V^\nu \) be a tensorial (or "covariant" derivative).

2. To be continued...