General Relativity HW8 Problems

1. Consider a unit 2-sphere with coordinates $(\theta, \phi)$ and metric $ds^2 = d\theta^2 + \sin^2 \theta d\phi^2$.
   a) Take a vector with components $V^\mu = (1, 0)$ and parallel transport it once around a circle of constant latitude. What are the components of the resulting vector as a function of the polar angle $\theta$ of the circle of constant latitude.
   b) Show that lines of constant longitude ($\phi =$ constant) are geodesics, and that the only line of constant latitude ($\theta =$ constant) that is a geodesic is the equator ($\theta = \frac{\pi}{2}$).

2. Okay, just to make sure we all ended up on the same page at the end of our discussion of geodesics and the twin paradox, consider the set of time-like paths in $\mathbb{R}^2$ (2D Minkowski space) that connect two points at the same spatial position and separated in time by some amount $\Delta t$ (according to coordinates adapted to some inertial observer). We argued in class that the geodesic path between these points is the spacetime path of maximal length. Argue that the set of all time-like paths (geodesic or otherwise) between the two points is bounded from below by a minimum length. Do this both geometrically (draw pictures) and from what you know about the invariant interval and time-like and light-like paths. What is different about this question compared to the case in $\mathbb{R}^2$ where we do not have both upper and lower bounds on paths between two points?

3. Consider the vector field in $\mathbb{R}^3$ that corresponds to the electric field of a unit point charge at the origin in spherical polar coordinates. You may ignore time in this problem. Find expressions in spherical polar coordinates for:
   a) The covariant derivative of the electric field.
   b) The directional covariant derivative of the electric field along the path given by $x^\mu(\lambda) = (r(\lambda), \theta(\lambda), \phi(\lambda)) = (\lambda^2, \frac{\pi}{2}, \lambda)$. 