General Relativity HW8 Problems

1. Consider the intermediate form of the metric we obtained when solving for the Schwarschild solution:

$$ds^2 = -A(r,t)dt^2 + 2B(r,t)drdt + C(r,t)dr^2 + r^2D(r,t)d\Omega^2$$

Suppose the function $D(r,t)$ ended up being of the form $D(r,t) = ar^2 + bt$ (where a and b have the right dimensions to that overall $D(r,t)$ is dimensionless). What we would do next is redefine the radial coordinate to be $r \to r'(r,t) = r\sqrt{D(r,t}$.

a. Using the explicit function given above, invert this transformation to find $r(r',t)$.

b. Plug this function into the expression $r^2D(r,t)$ and see what you get.

c. Suppose the function $A(r,t) = kr$. What form would this take after the transformation? I.e. what is $\tilde{A}(r',t)$. The point of this part is to help you realize that the functional dependence of $\tilde{A}$ on r' will be different than the functional dependence of A on r, hence the twiddle.

2. Consider Einstein's equations in a vacuum, but with cosmological constant Λ such that $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$.
   a) Solve for the most general spherically symmetric metric that reduces to the Schwarzchild metric when $\Lambda \to 0$.
   b) For the metric you derived, construct the effective radial potential for geodesic motion and plot the potential for massive particles with $L = 0$ for the three cases $\Lambda > 0, \Lambda = 0, \Lambda < 0$. Note: These are values of the cosmological constant, **not** the angular momentum.

3. Consider a perfect fluid in a static, circularly symmetric (2+1)-dimensional spacetime, equivalently, a cylindrical configuration in (3+1)-dimensions with perfect rotational symmetry.
   a) Show (don't prove) that the vacuum solution can be written as

   $$ds^2 = -dt^2 + \frac{1}{1 - 8GM}dr^2 + r^2d\theta^2$$

   where M is constant and $\theta \in [0,2\pi)$.
   b) Derive the analogue of the Tolman-Oppenhiemer-Volkoff equation for (2+1)-dimensions.
   c) Solve the (2+1)-dimensional TOV equation for a constant density star. Find $p(r)$ and solve for the metric.