General Relativity HW8 Problems

1. Consider the intermediate form of the metric we obtained when solving for the Schwarschild solution:

$$ds^2 = -A(r,t)dt^2 + 2B(r,t)drdt + C(r,t)dr^2 + r^2 D(r,t)d\Omega^2$$

Suppose the function $D(r,t)$ ended up being of the form $D(r,t) = ar^2 + bt$ (where a and b have the right dimensions to that overall $D(r,t)$ is dimensionless). What we would do next is redefine the radial coordinate to be $r \to r'(r,t) = r\sqrt{D(r,t}$.

a. Using the explicit function given above, invert this transformation to find $r(r',t)$.

b. Plug this function into the expression $r^2 D(r,t)$ and see what you get.

c. Suppose the function $A(r,t) = kr$. What form would this take after the transformation? I.e. what is $\tilde{A}(r',t)$. The point of this part is to help you realize that the functional dependence of $\tilde{A}$ on r' will be different than the functional dependence of A on r, hence the twiddle.

a) $r' = r\sqrt{ar^2+bt} \Rightarrow r'^2 = ar^4 + btr^2 \Rightarrow ar^4 + btr^2 - r'^2 = 0$

$r^2 = \dfrac{-bt \pm \sqrt{b^2t^2 + 4ar'^2}}{2a}$

use quadratic eqn. w/ r^2 as unknown

For $r^2 > 0$ we need the positive root in the expression above

$r = \sqrt{\dfrac{-bt + \sqrt{b^2t^2 + 4ar'^2}}{2a}}$

b) $r^2 D(r,t) = \dfrac{-bt + \sqrt{b^2t^2 + 4ar'^2}}{2a}\left[a\,\dfrac{-bt + \sqrt{b^2t^2+4ar'^2}}{2a} + bt\right]$

$= \dfrac{-bt + \sqrt{b^2t^2 + 4ar'^2}}{2a}\left[\dfrac{-bt + \sqrt{b^2t^2+4ar'^2}}{2} + bt\right]$

$= \dfrac{b^2t^2 + b^2t^2 + 4ar'^2 - 2bt\sqrt{b^2t^2+4ar'^2}}{4a} + \dfrac{-b^2t^2 + bt\sqrt{b^2t^2+4ar'^2}}{2a}$

$= \dfrac{b^2t^2 + 2ar'^2 - bt\sqrt{b^2t^2+4ar'^2}}{2a} + \dfrac{-b^2t^2 + bt\sqrt{b^2t^2+4ar'^2}}{2a}$

$= \dfrac{2ar'^2}{2a} = r'^2$ Boom!!

c) $A(r,t) = kr \Rightarrow A(r',t) = k\sqrt{\dfrac{-bt + \sqrt{b^2t^2 + 4ar'^2}}{2a}}$

which is definitely a different functional dependence on r' than just kr'!!

2. Consider Einstein's equations in a vacuum, but with cosmological constant Λ such that $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$.
   a) Solve for the most general spherically symmetric metric that reduces to the Schwarzchild metric when $\Lambda \to 0$.
   b) For the metric you derived, construct the effective radial potential for geodesic motion and plot the potential for massive particles with $L = 0$ for the three cases $\Lambda > 0, \Lambda = 0, \Lambda < 0$. Note: These are values of the cosmological constant, **not** the angular momentum.

a) $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ or $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0$

Tracing both sides: $R - 2R + 4\Lambda = 0$

$$R = 4\Lambda$$

Then our field equation becomes: $R_{\mu\nu} = \Lambda g_{\mu\nu}$

To find the most general spherically symmetric solution we can follow all of the steps we used to derive the Schwarzchild solution up until when we used the field equation ($R_{\mu\nu} = 0$ in that case).

Thus we start with:

$$ds^2 = -e^{2\alpha(r,t)} dt^2 + e^{2\beta(r,t)} dr^2 + r^2 d\Omega^2$$

Now we need the components of $R_{\mu\nu}$, but since the metric so far is the same as the Schwarzchild case, they will be the same. We only modify the r.h.s. of EE.

Then:

$$R_{tr} = \frac{2}{r}\frac{\partial \beta}{\partial t} = 0 \quad (\text{since } \Lambda g_{tr} = 0) \quad \Rightarrow \beta(r,t) = \beta(r)$$

$$R_{\theta\theta} = e^{-2\beta}\left[r\left(\frac{\partial \beta}{\partial r} - \frac{\partial \alpha}{\partial r} \right) - 1 \right] + 1 = \Lambda g_{\theta\theta} = \Lambda r^2$$

Again considering:

$$\partial_t R_{\theta\theta} = -2 \overset{0}{\cancel{\frac{\partial \beta}{\partial t}}} e^{-2\beta}\left[r\left(\frac{\partial \beta}{\partial r} - \frac{\partial \alpha}{\partial r} \right) - 1 \right] + e^{-2\beta}\left[\overset{0}{\cancel{r\frac{\partial^2 \beta}{\partial t \partial r}}} - r\frac{\partial^2 \alpha}{\partial t \partial r} \right]$$

$$= \partial_t(\Lambda r^2) = 0$$

So:

$$\frac{\partial^2 \alpha}{\partial t \partial r} = 0 \Rightarrow \alpha(r,t) = f(r) + g(t)$$

Then:

$$ds^2 = -e^{2f(r)} e^{2g(t)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2$$

Defining:

$$t' = \int e^{g(t)} dt \quad \Rightarrow dt' = e^{g(t)} dt$$

$$ds^2 = -e^{2f(r)} dt^2 + e^{2\beta(r)} + r^2 d\Omega^2$$ still same as Schwarzschild!

Then:

$$R_{tt} = e^{2(f-\beta)}\left[\frac{\partial^2 f}{\partial r^2} + \left(\frac{\partial f}{\partial r}\right)^2 - \frac{\partial f}{\partial r}\frac{\partial \beta}{\partial r} + \frac{2}{r}\frac{\partial f}{\partial r} \right] = \Lambda g_{tt} = -\Lambda e^{2f}$$

$$R_{rr} = -\frac{\partial^2 f}{\partial r^2} - \left(\frac{\partial f}{\partial r}\right)^2 + \frac{\partial f}{\partial r}\frac{\partial \beta}{\partial r} + \frac{2}{r}\frac{\partial \beta}{\partial r} = \Lambda g_{rr} = \Lambda e^{2\beta}$$

Consider:

$$e^{-2(f-\beta)} R_{tt} + R_{rr} = 0 \Rightarrow \frac{2}{r}\left[\frac{\partial f}{\partial r} + \frac{\partial \beta}{\partial r} \right] = 0$$

$$\Rightarrow f(r) = -\beta(r) + C$$

So far:

$$ds^2 = -e^{-2\beta(r)} \underbrace{e^{2c} dt^2}_{dt^2} + e^{2\beta(r)} dr^2 + r^2 d\Omega^2$$

Still same as Schwarzchild!

Finally:

$$R_{\theta\theta} = e^{2f}\left(-2r\frac{\partial f}{\partial r} - 1\right) + 1 = \Lambda g_{\theta\theta} = \Lambda r^2$$

or

$$\frac{\partial}{\partial r}\left(r e^{2f}\right) = 1 - \Lambda r^2 \Rightarrow r e^{2f} = \int [1 - \Lambda r^2] dr$$

$$= r - \frac{1}{3}\Lambda r^3 + c$$

$$e^{2f} = 1 - \frac{1}{3}\Lambda r^2 + \frac{c}{r} = e^{-2\beta}$$

$$ds^2 = -\left(1 + \frac{c}{r} - \frac{\Lambda r^2}{3}\right)dt^2 + \left(1 + \frac{c}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2$$

To agree w/ Schwarzchild when $\Lambda \to 0$ we need $c = -2GM$

$$ds^2 = -\left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)dt^2 + \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2$$

b) First note that the metric has no t or ϕ dependence.

Thus we immediately know 2 Killing vectors:

$$K^\mu = (1,0,0,0) \Rightarrow K_\mu = \left(-\left[1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right], 0, 0, 0\right)$$

$$R^\mu = (0,0,0,1) \Rightarrow R_\mu = (0,0,0,\underbrace{r^2 \sin^2\theta}_{= r^2 \text{ if we set } \theta = \frac{\pi}{2}}) = (0,0,0,r^2)$$

Then we have the conserved quantities:

i) $K_\mu \frac{dx^\mu}{d\lambda} = \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)\frac{dt}{d\lambda} \equiv E$

ii) $R_\mu \frac{dx^\mu}{d\lambda} = r^2 \frac{d\phi}{d\lambda} \equiv L$

Recall that for geodesics $\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$ is conserved.

iii) $\epsilon = \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2$

where $\epsilon = 0$ for $m_t = 0$ and $\epsilon = 1$ for $m_t > 0$ w/ $\lambda = \tau$.

Combining these three expressions (i-iii) we have:

$$\left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)\epsilon = E^2 - \left(\frac{dr}{d\lambda}\right)^2 - \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)\frac{L^2}{r^2}$$

or

$$\frac{1}{2}E^2 = \frac{1}{2}\left(\frac{\partial r}{\partial \lambda}\right)^2 + \underbrace{\frac{1}{2}\epsilon - \epsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} - \epsilon\frac{\Lambda r^2}{6} - \frac{\Lambda L^2}{6}}_{V_{eff}(r)}$$

For $m_t > 0$ w/ $\lambda = \tau$

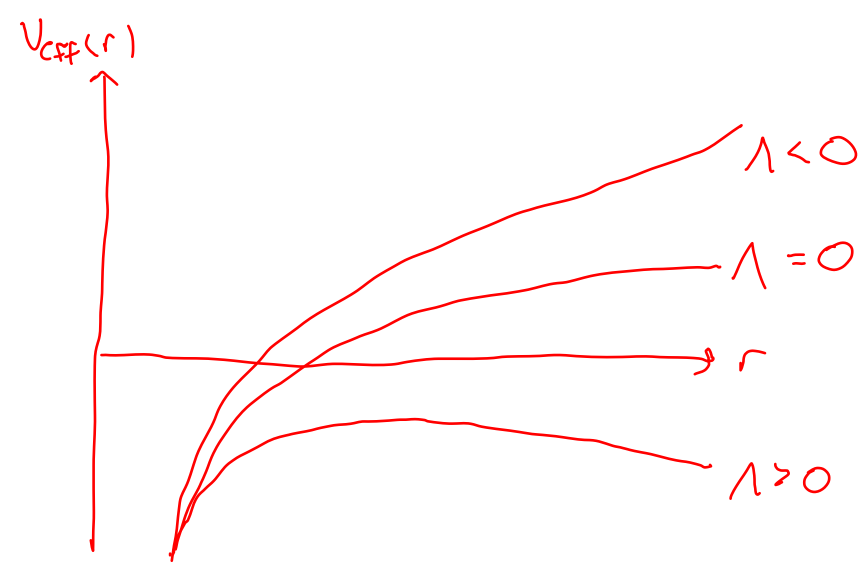
$\epsilon = 1$ and taking

$GM = 1$ and $L = 0$

for purely radial

motion we have:

$$V_{eff}(r) = \frac{1}{2} - \frac{1}{r} - \frac{\Lambda r^2}{6}$$



3. Consider a perfect fluid in a static, circularly symmetric (2+1)-dimensional spacetime, equivalently, a cylindrical configuration in (3+1)-dimensions with perfect rotational symmetry.

a) Show that the vacuum solution can be written as

$$ds^2 = -dt^2 + \frac{1}{1-8GM}dr^2 + r^2 d\theta^2$$

where M is constant and $\theta \in [0,2\pi)$.

b) Derive the analogue of the Tolman-Oppenhiemer-Volkoff equation for (2+1)-dimensions.

c) Solve the (2+1)-dimensional TOV equation for a constant density star. Find $p(r)$ and solve for the metric.

a) You can use Mathematica to verify that

$$g_{\mu\nu} = \begin{pmatrix} -1 & & \\ & (1-8GM)^{-1} & \\ & & r^2 \end{pmatrix}$$ w/ $\{t, r, \theta\}$ satisfies $R_{\mu\nu} = 0$

b) Consider a 2+1 D circular geometry that is <u>assumed</u> to be static, i.e.

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\theta^2$$

Using this metric and Mathematica we obtain:

$$G_{tt} = \frac{1}{r} e^{2\alpha(r) - 2\beta(r)} \frac{d\beta}{dr}$$

$$G_{rr} = \frac{1}{r}\frac{d\alpha}{dr}$$

$$G_{\theta\theta} = r^2 e^{-2\beta(r)}\left[\left(\frac{d\alpha}{dr}\right)^2 - \frac{d\alpha}{dr}\frac{d\beta}{dr} + \frac{d^2\alpha}{dr^2}\right]$$

Assuming a perfect fluid in its overall rest frame we have:

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} = \mathrm{diag}(e^{2\alpha}\rho, e^{2\beta}p, r^2 p)$$

or

$$T^{\mu\nu} = \mathrm{diag}(e^{-2\alpha}\rho, e^{-2\beta}p, r^{-2}p)$$ Note ρ, p depend on r.

Then $G_{tt} = 8\pi G T_{tt}$ becomes:

$$\frac{1}{r} e^{2\alpha - 2\beta} \frac{d\beta}{dr} = 8\pi G e^{2\alpha} \rho$$

$$e^{-2\beta} \frac{d\beta}{dr} = 8\pi G r \rho$$

And $G_{rr} = 8\pi G T_{rr}$ becomes:

$$\frac{1}{r} \frac{d\alpha}{dr} = 8\pi G e^{2\beta} p$$

$$\frac{d\alpha}{dr} = 8\pi G e^{2\beta} r p$$

Lastly we can use $\nabla_\mu T^{\mu\nu} = 0$ w/ $\nu = r$ to obtain:

$$\nabla_\mu T^{\mu r} = 0 = \nabla_t T^{tr} + \nabla_r T^{rr} + \nabla_\theta T^{\theta r}$$

$$= \partial_t T^{tr} + \Gamma^t_{t\lambda} T^{\lambda r} + \Gamma^r_{t\lambda} T^{t\lambda}$$

$$+ \partial_r T^{rr} + \Gamma^r_{r\lambda} T^{\lambda r} + \Gamma^r_{r\lambda} T^{r\lambda}$$

$$+ \partial_\theta T^{\theta r} + \Gamma^\theta_{\theta\lambda} T^{\lambda r} + \Gamma^r_{\theta\lambda} T^{\theta\lambda}$$

From Mathematica:

$$\Gamma^r_{rr} = \frac{d\beta}{dr} \quad \Gamma^r_{\theta\theta} = -e^{-2\beta} r \quad \Gamma^r_{tt} = \frac{d\alpha}{dr} e^{2\alpha - 2\beta} \quad \Gamma^\theta_{\theta r} = \frac{1}{r} \quad \Gamma^t_{tr} = \frac{d\alpha}{dr}$$

Using these and keeping only nonzero terms (remember $T^{\mu\nu}$ is diagonal):

$$\nabla_\mu T^{\mu\nu} = \partial_r T^{rr} + 2\Gamma^r_{rr} T^{rr} + \Gamma^\theta_{\theta r} T^{rr} + \Gamma^r_{\theta\theta} T^{\theta\theta} + \Gamma^t_{tr} T^{rr} + \Gamma^r_{tt} T^{tt}$$

$$= \frac{d}{dr}(e^{-2\beta} p) + 2\frac{d\beta}{dr} e^{-2\beta} p + \cancel{\frac{1}{r} e^{-2\beta} p} \cancel{- e^{-2\beta} \frac{1}{r} p} + \frac{d\alpha}{dr} e^{-2\beta} p + \frac{d\alpha}{dr} e^{-2\beta} \rho$$

$$= 0$$

Simplifying:

$$0 = \frac{dp}{dr} + \frac{d\alpha}{dr}(\rho + p) \quad \Rightarrow \quad \frac{dp}{dr} = -\frac{d\alpha}{dr}(\rho + p)$$

Now we make the "inspired" replacement:

$$e^{-2\beta(r)} = 1 - 8GM(r) \quad \Rightarrow \frac{d}{dr}\left(e^{-2\beta}\right) = -2\frac{d\beta}{dr}e^{-2\beta}$$

$$\text{or} \quad \frac{d\beta}{dr}e^{-2\beta} = -\frac{1}{2}\frac{d}{dr}\left(1 - 8GM(r)\right) = 4G\frac{dM}{dr}$$

Then $G_{tt} = 8\pi G T_{tt}$ becomes:

$$4G\frac{dM}{dr} = 8\pi G r\rho \quad \Rightarrow \quad \frac{dM}{dr} = 2\pi r\rho \quad \Rightarrow \quad M(r) = 2\pi\int_0^r r'\rho(r')dr'$$

Combining all 3 eqns. and eliminating α and β yields:

$$8\pi G r\left(1 - 8GM(r)\right)^{-1} p = -\frac{1}{(\rho + p)}\frac{dp}{dr}$$

or

$$\frac{dp}{dr} = -\frac{(\rho + p)\,8\pi G r p}{1 - 8GM(r)} \qquad \text{The 2+1 D T.O.V. eqn.}$$

c) Assuming constant density, i.e. $\rho(r) = \begin{cases} \rho_* & r \leq R \\ 0 & r > R \end{cases}$

we have:

$$M(r) = 2\pi\int_0^r r'\rho_* dr' = \pi r^2 \rho_* \qquad r \leq R$$

then:

$$\frac{dp}{dr} = -\frac{(\rho_* + p)\,8\pi G r p}{1 - 8\pi G\rho_* r^2} \quad \Rightarrow \quad \frac{dp}{(\rho_* + p)p} = -\frac{8\pi G r\,dr}{1 - 8\pi G\rho_* r^2}$$

Integrating both sides:

$$-\frac{1}{\rho_*}\ln\left(\frac{\rho_*}{p}+1\right) = \frac{1}{2\rho_*}\ln\left(1-8\pi G\rho_* r^2\right) + c$$

or

$$\ln\left(\frac{\rho_*}{p}+1\right)^{-1} = \ln\left(1-8\pi G\rho_* r^2\right)^{1/2} + c$$

$$\frac{\rho_*}{p} = \frac{1}{e^c\sqrt{1-8\pi G\rho_* r^2}} - 1$$

$$p(r) = \frac{\rho_*}{\frac{1}{e^c\sqrt{1-8\pi G\rho_* r^2}} - 1} = \frac{\rho_* e^c\sqrt{1-8\pi G\rho_* r^2}}{1-e^c\sqrt{1-8\pi G\rho_* r^2}}$$

In terms of $M = \pi R^2\rho_*$ this becomes:

$$p(r) = \frac{\rho_* e^c\sqrt{1-8GM\left(\frac{r}{R}\right)^2}}{1-e^c\sqrt{1-8GM\left(\frac{r}{R}\right)^2}}$$

Now we need to make sure that $p(R)=0$ (the pressure should vanish outside of the source).

$$p(R) = \frac{\rho_* e^c\sqrt{1-8GM}}{1-e^c\sqrt{1-8GM}}$$

But no choice of c (except $c=-\infty$) does this. But note that if $1=8GM$ then $p(R)=0$ for any c!

We have found the odd result that in 2+1D, hydrostatic equilibrium of a circularly symmetric source requires $8GM=1$. This result can be shown to hold for any perfect fluid w/ a polytropic equation of state, i.e. $p = k\rho^{1+\frac{1}{n}}$ which encompasses a large array of sources.