So far we have been working with derivatives.

Covariant Derivative: \( \frac{\partial}{\partial \lambda} V^\mu = \lim_{\lambda \to 0} \frac{V^\mu(x^\lambda + \lambda)}{\lambda} \)

Directional Covariant Derivative: \( \frac{\partial}{\partial \lambda} V^\mu = \partial \lambda \phi V^\nu \frac{\partial}{\partial \lambda} \frac{\partial V^\mu}{\partial \phi} \) = \frac{d}{d\lambda} \phi V^\nu \frac{\partial}{\partial \phi} V^\mu

Both use \( \phi \)-transport so both are covariant. The difference is the path.

In \( \frac{\partial}{\partial \lambda} V^\mu \) we pick a value of \( \mu \) (a coordinate) and the path is a shift along \( x^\mu \).
In \( \frac{\partial}{\partial \phi} V^\nu \) we shift along the curve \( x^\phi(\lambda) \) which can be arbitrary, i.e. not aligned along coordinate axes! Of course if we know how the curve changes w/ coordinates \( \frac{dx^\phi}{d\lambda} \), and we know how the vector changes w/ coordinates \( \frac{\partial}{\partial \phi} V^\mu \), then we can combine these to determine how the vector changes along the curve \( \frac{d}{d\lambda} \phi V^\nu \), i.e. \( \frac{d}{d\phi} \frac{d}{d\lambda} \phi V^\nu \).
Geodesics

Recall: E & h (Maxwell + Lorentz Form of Newton) GR (EE + Geodesics)

There are 2 ways to define geodesic paths $x^g(\lambda)$:
1) Curves $x^g(\lambda)$ which extremize the distance between two points.
2) Curves $x^g(\lambda)$ which $\parallel$-transport their own tangent vectors.

In $\mathbb{R}^2$:

![Diagram showing geodesics in $\mathbb{R}^2$]

We know that straight lines are the shortest paths and we can see that the tangent vectors are $\parallel$-transported.

Consider a non-geodesic path in $\mathbb{R}^2$:

![Diagram showing a non-geodesic path in $\mathbb{R}^2$]

Note: Tangent vectors are not $\parallel$!

To really drive the point home consider $\parallel$-transporting a vector along a circle in different spaces:

In $\mathbb{R}^2$:

![Diagram showing $\parallel$-transporting along a circle in $\mathbb{R}^2$]

Note: What started as a tangent vector is no longer tangent!

But notice:

Not the shortest distance between $A$ & $B$!

In $S^1$:

![Diagram showing $\parallel$-transporting along a circle in $S^1$]

Note: Tangent to $x^g(\lambda)$

Still tangent to $x^g(\lambda)$

The shortest distance between $A$ & $B$!
To formalize the definition of geodesics by $\parallel$-transport recall that a curve $\gamma^\alpha(\lambda)$ has components $\frac{d\gamma^\alpha}{d\lambda}$ of its tangent vector.

If $\gamma^\alpha(\lambda)$ is a geodesic (call it $\gamma^\alpha_{\parallel}(\lambda)$) then these components should be covariantly constant along the curve, i.e. $\parallel$ to each other.

Then: $\gamma^\alpha(\lambda)=\gamma^\alpha_{\parallel}(\lambda)$ if \[ \frac{D}{d\lambda} \frac{d\gamma^\alpha}{d\lambda} = 0 = \frac{d\gamma^\alpha}{d\lambda} \nabla_{\frac{d\gamma^\alpha}{d\lambda}} \frac{d\gamma^\alpha}{d\lambda} = \frac{d\gamma^\alpha}{d\lambda} \left( \Gamma^\alpha_{\nu\mu} \frac{d\gamma^\nu}{d\lambda} \frac{d\gamma^\mu}{d\lambda} \right) \]

The geodesic equation

\[ \frac{d^2\gamma^\alpha}{d\lambda^2} + \Gamma^\alpha_{\nu\mu} \frac{d\gamma^\nu}{d\lambda} \frac{d\gamma^\mu}{d\lambda} = 0 \]

To use this first note that it is 2nd order so we need 2 boundary conditions before solving. We could give an initial position $\gamma^\alpha(\lambda_0)$ and "velocity" $\frac{d\gamma^\alpha}{d\lambda}_{\lambda_0}$ and then this generates the geodesic "launched" from there.

Alternatively, and more familiar, we could give an initial and final position and this gives the extremal path between them.

Since $\Gamma^\alpha_{\nu\mu}$ depends on $\gamma$, the explicit form will vary for different geodesics.

An intuitive example: $\mathbb{R}^3 \ni (x,y,z) \Rightarrow \Gamma = 0 \Rightarrow \frac{d\gamma^\alpha}{d\lambda} = 0$ for geodesics

$\Downarrow$

$\gamma(\lambda) = \lambda \Sigma^\alpha + \gamma^{\alpha}_{\parallel}$

constants set by boundary conditions

A straight line! Clearly the shortest path in $\mathbb{R}^3$. 

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To see the extremization more explicitly consider another example.

\[ \bar{L} = \int \sqrt{g} \, ds = \int ds + r \, \phi(\phi) \rightarrow \Gamma^\phi_{\theta\theta} = -r, \quad \Gamma^\phi_{\theta r} = \Gamma^\phi_{r\theta} = \frac{1}{r} \]

\[ X^\lambda(x) = (\phi(x), \theta(x)) \]

Parameterize with \( s = \text{distance along the curve} \), i.e., \( X^\lambda(s) = (\phi(s), \theta(s)) \).

Then the total length is:

\[ L = \int A \sqrt{g_{\lambda\lambda}} \, ds = \int A \sqrt{g_{\phi\phi} + \frac{r^2}{r^2 + 1} \, ds} \]

Extremizing this is akin to extremizing an action \( S = \int L(x, \nabla x) \, ds \) in \( Cl \).

Normally:

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{X}} \right) = \frac{\partial L}{\partial X} \]

If we call \( \frac{d\lambda}{ds} = V_\lambda \), \( \frac{d\phi}{ds} = V_\phi \) then:

\[ \frac{d}{ds} \left( \frac{\partial L}{\partial V_\phi} \right) - \frac{\partial L}{\partial \phi} = \frac{d^2}{ds^2} \frac{\partial L}{\partial \phi} + \frac{1}{r} \frac{d}{ds} \frac{\partial L}{\partial \phi} = 0 \]

Compare this to the geodesic equation:

\[ \frac{d^2}{ds^2} + \Gamma^\lambda_{\phi\theta} \frac{d\phi}{ds} \frac{d\lambda}{ds} + \Gamma^\lambda_{\theta\phi} \frac{d\theta}{ds} \frac{d\lambda}{ds} = \frac{d^2 \phi}{ds^2} + \frac{1}{r} \frac{d\phi}{ds} \frac{d\phi}{ds} = 0 \]
If we worked in a Lorentzian signature space, e.g., Minkowski, then $\gamma_{\alpha}=\int^{\theta}_{0}\sqrt{-dt^{2}}$ and timelike geodesics actually minimize the spacetime length.

To appreciate this consider a geodesic (constant velocity or at rest) object $U$ in $M^{4}$ and an accelerated non-geodesic object $V$. 

For $U$: $S_{U} = \int^{t_{2}}_{t_{1}} dt = t_{2} - t_{1}$

For $V$: $S_{V} = \int^{t_{2}}_{t_{1}} \sqrt{-dt^{2}} = \int^{t_{2}}_{t_{1}} (1 - v^{2}) dt = \int^{t_{2}}_{t_{1}} (1 - v_{0}^{2}) dt + \int^{t_{2}}_{t_{1}} (1 - (v(t) - v_{0})^{2}) dt$

\[ S_{V} < t_{2} - t_{1} \]

B. T. W. These two trajectories are representative of those in the twin paradox. The twin who remains on Earth follows $U$ and is older than the one that travels away and then back along $V$. 