Something that will prove useful is to compute the commutator of covariant derivatives,
\[ [\nabla_\alpha, \nabla_\mu] V^\lambda = \nabla_\alpha \nabla_\mu V^\lambda - \nabla_\mu \nabla_\alpha V^\lambda \]
\[ = \partial_\alpha \left( \nabla_\mu V^\lambda \right) - \nabla_\alpha \nabla_\mu V^\lambda + \Gamma^\lambda_{\mu\kappa} \nabla_\kappa V^\lambda - (\text{underlined terms}) \]
\[ = \partial_\alpha \partial_\mu V^\lambda + \left( \partial_\alpha \Gamma^\lambda_{\mu\beta} \right) V^\beta + \underbrace{\Gamma^\lambda_{\mu\kappa} \partial_\kappa V^\lambda}_{\text{underlined terms}} - \Gamma^\lambda_{\kappa\mu} \partial_\kappa V^\lambda \]
\[ - \Gamma^\lambda_{\mu\nu} \Gamma^\mu_{\alpha\nu} V^\beta + \Gamma^\lambda_{\kappa\mu} \partial_\kappa V^\lambda + \Gamma^\lambda_{\mu\kappa} \Gamma^\kappa_{\nu\lambda} V^\beta - (\text{underlined terms}) \]

The underlined terms will cancel when combined with (underlined terms) subtraction, leaving:
\[ [\nabla_\alpha, \nabla_\mu] V^\lambda = (\partial_\alpha \Gamma^\lambda_{\mu\beta} - \partial_\mu \Gamma^\lambda_{\alpha\beta} + \Gamma^\mu_{\kappa\lambda} \partial_\kappa V^\lambda - \Gamma^\kappa_{\mu\lambda} \partial_\kappa V^\lambda) V^\beta + 2 \Gamma^\lambda_{\mu\nu} \nabla_\nu V^\lambda \]

Recall \( \Gamma \sim \hbar \Rightarrow \quad \frac{\delta^2 G}{\delta g^2} \quad \text{(underlined terms)} \)

Good measures of curvature (do not vanish even in LICs!)!

Secretly we just computed something incredibly important, but more on that next time!
**Flatness**

We would like a coordinate invariant way of saying a space is flat.

1. We could look at the metric, but even in flat space the metric can look funny if we choose the wrong coordinates.

   Recall $g_{uv} \to g'_{uv} = \frac{\partial x^m}{\partial x^u} \frac{\partial x^n}{\partial x^v} g_{mn}$ so even if $g_{uv} = (\delta_{ij})$ we can get crazy $g'_{uv}$.

2. We do know that in flat space w/ cartesian coordinates $\Gamma^m_{uv} = 0$, so we could say this is a condition for flatness.

   *Note: This might work if $\Gamma$ was a tensor since $\Gamma^m_{uv} = \frac{\partial x^m}{\partial x^u} \frac{\partial x^m}{\partial x^v} \Gamma^m_{uv}$

   But it **is not**:

   $$\Gamma^m_{uv} = \frac{\partial x^m}{\partial x^u} \frac{\partial x^m}{\partial x^v} \Gamma^m_{uv} + \frac{\partial x^m}{\partial x^v} \frac{\partial x^m}{\partial x^u} \Gamma^m_{uv}$$

   **not zero!**

3. Also remember that at any point we can always choose LCC’s so that $g_{uv} = (\delta_{ij})$ and $\Gamma = 0$, even if the space is curved!

What are we to do?!
Recall:

If we transport any vector around a closed path in curved space then it generally comes back changed.

Okay, so let's choose an infinitesimal closed path: \( \Delta x^a \)

We could start with a vector at \( A \) and then transport it:

\( (-\Delta x^\nu)(-\Delta x^\mu) \Delta x^\mu \Delta x^\nu \mathcal{V}_A = \mathcal{V}_B \) and then compute \( \mathcal{W}_\nu - \mathcal{V}_\nu = \mathcal{S}_\nu \) (=0 for flat)

Or we could take the vector from \( A \) to \( B \) along two paths:

\( \Delta x^\nu \Delta x^\mu \mathcal{V}_A = \mathcal{V}_B \), \( \Delta x^\mu \Delta x^\nu \mathcal{V}_A = \mathcal{V}_B \) then compute \( \mathcal{W}_\nu - \mathcal{V}_\nu = \mathcal{S}_\nu \) (=0 for flat)

But this is exactly what \( \mathcal{L}_{\mathcal{A}} \), \( \nabla \mathcal{A} \) does!! (Remember that guy?)
\[ [\nabla_\mu, \nabla_\nu] \nabla^\lambda = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu + \Gamma^\lambda_{\mu\kappa} \Gamma^\kappa_\nu - \Gamma^\lambda_{\nu\kappa} \Gamma^\kappa_\mu ) \nabla^\lambda = R^\lambda_{\mu\nu} \]

But it gets better because this thing is a tensor, because we are just taking “good” derivatives of tensors.

In 4D this could have \( 1^4 = 256 \) independent components.

But we can show (by first forming \( R^\lambda_{\mu\nu} = g_{\kappa\lambda} R^\kappa_{\mu\nu} \)):

\[
\begin{align*}
R^\lambda_{\mu\nu} &= -R^\lambda_{\nu\mu} \quad \text{(antisymmetry in last 2, which follows from } [\nabla_\mu, \nabla_\nu]) = -[\nabla_\nu, \nabla_\mu]) \\
R^\lambda_{\mu\nu} &= -R^\lambda_{\nu\mu} \quad \text{(antisymmetry in first 2)} \\
R^\lambda_{\mu\nu} &= R^\lambda_{\nu\mu} \quad \text{(symmetry under } 2 \leftrightarrow 3) \\
R^\lambda_{\mu\nu} + R^\lambda_{\nu\mu} + R^\lambda_{\mu\nu} &= 0 \quad \text{(cyclic reorder of last 3)}
\end{align*}
\]

All of these symmetries reduce \( 256 \rightarrow 120 \) independent components.
If $\check{R}^{\lambda}{}_{\rho\mu\nu} = 0$, then the space is flat. Note, since $\check{R}^{\lambda}{}_{\rho\mu\nu}$ is a tensor, this is coordinate invariant, i.e. \[
\frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x^\rho} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \check{R}^{\lambda}{}_{\rho\mu\nu} = 0 \quad \Rightarrow \quad \check{R}^{\lambda}{}_{\rho\mu\nu} = 0
\]

There are other degrees of flatness which can be specified via objects built from $\check{R}^{\lambda}{}_{\rho\mu\nu}$.

First a quick useful fact: \[\check{\check{T}}^{(\rho\nu)} W_{\lambda\alpha\mu} = 0\]

2D example: \[\check{\check{T}}^{(\rho\nu)} W_{\lambda\alpha\mu} = \check{T}^{00} W_{00} + \check{T}^{01} W_{01} + \check{T}^{10} W_{10} + \check{T}^{11} W_{11} \Rightarrow \check{T}^{00} W_{00} = 0, \quad -\check{T}^{01} W_{01} = 0\]

Consider all 4-index contractions of $\check{R}^{\lambda}{}_{\rho\mu\nu}$:

1) $\check{R}^{\lambda}{}_{\rho\nu\mu} = \check{g}_{\lambda\mu} R^{\rho\nu} = \check{g}^{(\lambda\mu) \check{R}_{\nu}} = 0$

2) $\check{R}^{\rho}{}_{\nu\lambda\mu} = \check{g}_{\rho\lambda} \check{R}^{\nu}{}_{\mu} = \check{R}^{\nu}{}_{\mu} \quad \text{Ricci Tensor}$

3) $\check{R}^{\lambda}{}_{\rho\nu\mu} = -\check{R}^{\mu}{}_{\rho\nu\lambda} = -\check{R}^{\mu}{}_{\rho\lambda\nu} \quad \text{(same as Ricci Tensor)}$

Note: $\check{R}_{\rho\mu} = \check{R}_{\mu\rho} \quad \text{since} \quad \check{R}_{(\lambda\alpha\mu\lambda)}$.

Ricci is symmetric.

We can then form: \[\check{R} = \check{R}^{\nu}{}_{\nu} = \check{g}^{\nu\mu} \check{R}_{\mu\nu} \quad \text{Ricci Scalar}\]

And then: $\check{R}^{\lambda}{}_{\rho\mu\nu} - \text{all contractions} = \check{C}_{\rho\mu\nu} \quad \text{Weyl Tensor}$

Then \{ $\check{C}_{\rho\mu\nu}$, $\check{R}_{\mu\nu}$, $\check{R}$ \} contains everything in $\check{R}^{\lambda}{}_{\rho\mu\nu}$. 

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How flat is flat?

<table>
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<th>$\nabla^2 R_{\alpha\beta} = 0$</th>
<th>Flat - Flat</th>
<th>Examples</th>
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<tr>
<td>$R_{\alpha\beta} = 0$</td>
<td>Ricci-flat</td>
<td>$\mathbb{R}^4$, $\mathbb{T}^4$</td>
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<tr>
<td>$C_{\alpha\beta\gamma\delta} = 0$</td>
<td>Conformally-flat</td>
<td>All 4D pseudo-Riemannian manifolds (since $C_{\alpha\beta\gamma\delta} = 0$) Can be mapped to flat space or conformal transformation (locally angle preserving trans.)</td>
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<tr>
<td>$R = 0$</td>
<td>Doesn't mean much (however for certain maximally symmetric spaces, $R$ completely determines the curvature.)</td>
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