Now that we have the basic content of GR down, we should recall that this theory is supposed to generalize (be a relativistic version of) Newtonian gravity.

So what is Newton?

Recall the 2 parts of GR:

a) How does space get curved? Einstein's equation \( R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = 8 \pi G T_{\mu \nu} \)

b) How does curved space influence motion? Geodesic Equation \( \frac{\partial^2 x^\mu}{\partial \tau^2} + \Gamma^\mu_{\alpha \beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \)

These each have a non-relativistic limit in the Newtonian theory:

a) \( \varphi = -\frac{1}{2} \frac{G}{c^2} \rho \) where \( \varphi \) is the gravitational potential (hence \( \ddot{\varphi} = -\ddot{\varphi} \)) and \( \rho \) is mass density.

b) \( \ddot{x} = -\ddot{\varphi} \) which is just Newton's second law of gravity after cancelling masses.

We will extract each of these Newtonian results from the corresponding equations in GR using the following limits:

1) Small velocities \( \frac{dx}{d\tau} \ll c = 1 \) or \( \frac{dx}{d\tau} \ll \frac{dx^\alpha}{d\tau} = \frac{dx^\alpha}{dt} \) (recall: \( \alpha \) is a spatial index).

2) Weak gravitational field \( \tilde{g}_{\mu \nu} = g_{\mu \nu} + \Gamma_{\mu \nu} \) with \( \Gamma_{\mu \nu} \ll 1 \) (so ignore terms \( O(1) \) and higher).

3) Static gravitational field \( \tilde{g}_{\mu \nu} = 0 \) (let's avoid gravito-magnetic effects).

Due to 2), we will need an inverse metric which is: \( g^{\mu \nu} = \eta^{\mu \nu} - \eta^{\mu \alpha} \eta^{\nu \beta} h_{\alpha \beta} \) 

To see that this is the right choice consider: \( g_{\mu \nu} g^{\nu \rho} = \delta^\rho_{\mu} \)

\[ = (\eta_{\alpha \mu} + h_{\alpha \mu})(\eta^{\alpha \nu} - \eta^{\alpha \alpha} \eta^{\nu \beta} h_{\alpha \beta}) \]

\[ = \eta_{\alpha \mu} \eta^{\alpha \nu} - \eta_{\alpha \alpha} \eta^{\alpha \nu} h_{\alpha \beta} + \eta^{\nu \beta} h_{\alpha \alpha} + \delta^\rho_{\mu} \]

\[ = \delta^\rho_{\mu} - \delta^\rho_{\mu} h_{\alpha \beta} + \eta^{\nu \beta} h_{\alpha \alpha} \]

\[ = \delta^\rho_{\mu} \]
Let's start by looking for \( \zeta = -\dot{\Phi} \). We have:

\[ \frac{d^2 \zeta}{d t^2} + \gamma_{\mu \nu} \frac{d \zeta}{d x^\nu} \frac{d \zeta}{d x^\mu} = 0. \]

Consider a massive particle and let \( \lambda = t \) (proper time). Then:

\[ \frac{d^2 \zeta}{d t^2} + \gamma_{\mu \nu} \frac{d \zeta}{d x^\nu} \frac{d \zeta}{d x^\mu} = 0 = \frac{d^2 \zeta}{d t^2} + \gamma_{00} \frac{d \zeta}{d x^0} \frac{d \zeta}{d x^0} \quad \text{using (a)} \]

Using (b):

\[ \gamma_{00} \frac{d \zeta}{d x^0} \frac{d \zeta}{d x^0} = -\frac{1}{2} \left( \gamma_{\lambda \mu} \gamma_{\lambda \nu} \gamma_{\mu \nu} \right) \partial \lambda \zeta \partial \lambda \zeta = -\frac{1}{2} \gamma_{00} \partial_t \zeta \partial_t \zeta \quad \text{by (c)} \]

\[ = -\dot{\zeta} \partial_t \zeta \quad \text{by (b)} \]

Equations (one for each \( \mu \)):

\[ \frac{d^2 \zeta}{d t^2} - \frac{1}{2} \gamma_{\mu \nu} \partial \lambda \zeta \partial \lambda \zeta \left( \frac{d \zeta}{d x^0} \right)^2 = 0 \]

We have \( \partial_t \zeta = 0 \) which means we can take \( t = \lambda \). This is good because we know that in non-relativistic physics we can adequately parameterize motion of coordinate time \( t \).

\[ \mu = 0 \quad \frac{d^2 \zeta}{d t^2} - \frac{1}{2} \partial_t \zeta \partial_t \zeta \left( \frac{d \zeta}{d x^0} \right)^2 = 0 = \frac{d^2 \zeta}{d t^2} + \frac{1}{2} \partial_t \zeta \partial_t \zeta \left( \frac{d \zeta}{d x^0} \right)^2 \]

Only \( \partial_t \zeta = 0 \) is zero.

\[ \text{If we identify this as } \phi = -\frac{1}{2} \zeta \partial_t \zeta \]

then \( \dot{\phi} = -\frac{1}{2} \zeta \partial_t \zeta \).
Now we look for \( \phi' \phi = 4 \pi G \rho \) in \( R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = 8 \pi G T_{\mu \nu} \).

First we rewrite Einstein's eqn. in trace-reverse form.

Trace our both sides:

\[
\begin{align*}
\frac{q}{\mu \nu} (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) &= g_{\mu \nu} (8 \pi G T_{\mu \nu}) \\
R &= -\frac{1}{2} R = 8 \pi G T \\
R &= -8 \pi G T
\end{align*}
\]

Therefore:

\[
\begin{align*}
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} (-8 \pi G T) &= 8 \pi G T_{\mu \nu} \\
R_{\mu \nu} &= 8 \pi G (T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T) \quad \text{Trace-reverse form of EE}
\end{align*}
\]

We need to specify a source. In Newtonian gravity only mass contributes to gravity (not energy, momentum, pressure, etc.) so we take:

\[
T_{\mu \nu} = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix} \quad \text{(Perfect fluid in its rest frame)}
\]

Then:

\[
\begin{align*}
T &= g_{\mu \nu} T_{\mu \nu} \\
&= (\rho - \frac{1}{2} \delta_{\mu \nu} \rho) \delta_{\mu \nu} \quad \text{since only } T_{00} = \rho \neq 0 \\
&= (-1 - h_{00}) \rho
\end{align*}
\]

So the r.h.s. of trace-reverse EE becomes:

\[
8 \pi G (T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T) = 8 \pi G \left[ T_{00} - \frac{1}{2} g_{00} (-1-h_{00}) \rho \right] \\
= 8 \pi G \left[ \rho - \frac{1}{2} \rho + \Theta (\rho) \right] \\
= 4 \pi G \rho
\]

For the l.h.s. we have:

\[
\begin{align*}
R_{00} &= R_{00}^\lambda \lambda = \frac{\partial}{\partial x^0} (\frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^0}) \frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^0} \frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^0} \frac{\partial}{\partial x^0} = \phi''
\end{align*}
\]

Then:

\[
\phi'' = 4 \pi G \rho \quad \text{BAH!!}
\]