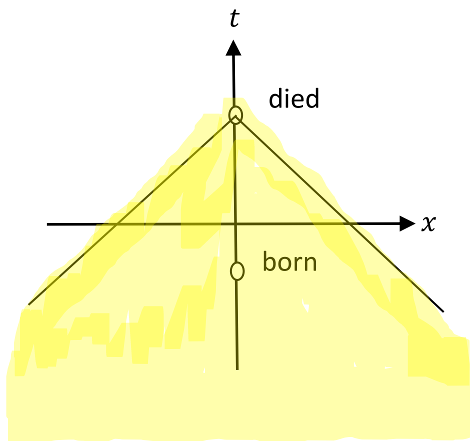
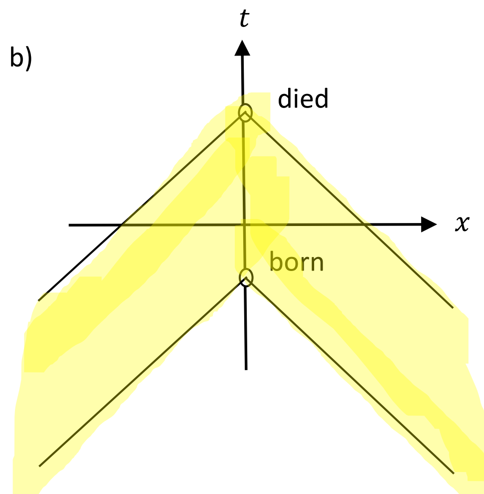
General Relativity Midterm Exam Name____Solutions________

1. Start in $\mathbb{M}^2$, i.e. 2D Minkowski spacetime, with coordinates $\{t, x\}$. Consider an incredibly boring lifetime wherein a person is born and then dies without ever changing their spatial position.
   a) On the first spacetime diagram provided below, pick and label two events that could correspond to their birth and death, and then indicate what region of the spacetime diagram could influence their life, i.e. what events are causally connected to the events of their life?
   b) On the second spacetime diagram, using the same two events of birth and death, indicate what region of events they might be able to "see" in their lifetime.

a)



b)



   c) How many independent transformations will leave the metric invariant?
   The spacetime is maximally symmetric, so we should expect three, since in 2D we would have $\frac{1}{2}2(2+1) = 3$ independent transformations which are symmetries of the spacetime. Of course these are just the shifts in t and x, and then a boost that mixes t and x.

2. Now consider the coordinate changes $u = e^t$ and $v = 2x$.
   a) Given that $t, x \in (-\infty, \infty)$, what are the ranges of u and v? $u \in (0, \infty), v \in (-\infty, \infty)$
   b) What is the metric for the space in the coordinates $\{u, v\}$? Hint: There is more than one way to do this.

   First of all note that $t = \ln(u)$ and $x = \frac{1}{2}v$.

   1. The metric transforms as $g_{\mu\nu} \to g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} = \frac{\partial x^\mu}{\partial x^{\mu'}} g_{\mu\nu} \frac{\partial x^\nu}{\partial x^{\nu'}} =$
   $$\begin{pmatrix} \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{1}{u} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{u} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{u^2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}.$$
   2. Or we can simply take the line interval $ds^2 = -dt^2 + dx^2$ and transform it using $dt = \frac{1}{u}du$ and $dx = \frac{1}{2}dv$ to obtain $ds^2 = -\frac{1}{u^2}du^2 + \frac{1}{4}dv^2$.

c) If a dual-vector has components $A_\mu = (1,1)$ in the $\{u, v\}$ coordinate system, then what are the components of the corresponding vector A^μ? We have $A^\mu = g^{\mu\nu}A_\nu$ where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, i.e. $g^{\mu\nu} = \begin{pmatrix} -u^2 & 0 \\ 0 & 4 \end{pmatrix}$. Therefore $A^\mu = \begin{pmatrix} -u^2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -u^2 \\ 4 \end{pmatrix}$.

d) What are the nonzero Christoffel connection coefficients for this space in the $\{u, v\}$ coordinate system? We can use Mathematica, or do it by hand using the expressions

$$\Gamma^v_{uu} = -\frac{1}{2}(g_{vv})^{-1}\partial_v g_{uu} = 0, \qquad \Gamma^u_{vv} = -\frac{1}{2}(g_{uu})^{-1}\partial_u g_{vv} = 0$$

$$\Gamma^v_{uv} = \partial_u\left(ln\sqrt{|g_{vv}|}\right) = 0, \qquad \Gamma^u_{vu} = \partial_v\left(ln\sqrt{|g_{uu}|}\right) = 0$$

$$\Gamma^u_{uu} = \partial_u\left(ln\sqrt{|g_{uu}|}\right) = \partial_u \ln\left(\frac{1}{u}\right) = -\frac{1}{u}, \qquad \Gamma^v_{vv} = \partial_v\left(ln\sqrt{|g_{vv}|}\right) = 0$$

e) Is the spacetime described by $\{u, v\}$ flat? How do you know? Hint: There is more than one way to do this.

1. First of all we could calculate the Riemann tensor using Mathematica, or by hand, and we would find that all of its components are zero.
2. On the other hand, we could argue that if we started with the flat spacetime $\mathbb{M}^2$ and just did a coordinate redefinition, then the tensor properties of the spacetime itself shouldn't change, e.g. its curvature should remain the same, zero.

f) How many independent Killing vectors does this spacetime have? Explain. Hint: There is more than one way to do this.

1. First of all, just like in the previous argument, since Killing's equation is a tensor equation, then changing coordinates should not change the number of solutions. Since $\mathbb{M}^2$ has three solutions, so too does the spacetime described in $\{u, v\}$.
2. Or we can argue that since the Riemann tensor is zero, then so is the Ricci scalar, and therefore the condition for a maximally symmetric spacetime, i.e. $R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})$, and hence there must be three Killing vectors.
3. Or we can show this explicitly. Consider the Killing vector equation $\nabla_{(\mu}K_{\nu)} = 0$. This will have three independent components given by:

$$\nabla_{(u}K_{u)} = \partial_u K_u - \Gamma^u_{uu}K_u - \Gamma^v_{uu}K_v = \partial_u K_u + \frac{1}{u}K_u = 0$$
$$\nabla_{(v}K_{v)} = \partial_v K_v - \Gamma^u_{vv}K_u - \Gamma^v_{vv}K_v = \partial_v K_v = 0$$
$$\nabla_{(u}K_{v)} = \partial_u K_v - \partial_v K_u - \Gamma^u_{uv}K_u + \Gamma^v_{uv}K_v = \partial_u K_v + \partial_v K_u = 0$$

With the three solutions:

$$K_\mu = (0,1), \quad K_\mu = \left(\frac{1}{u}, 0\right), \quad K_\mu = (-\tfrac{1}{2}v, \ln u)$$

g) Is the path $\{u(\lambda), v(\lambda)\} = \{\lambda, \lambda\}$ a geodesic? Prove your answer.

The geodesic equation is $\frac{\partial^2 x^\mu}{\partial\lambda^2} - \Gamma^\mu_{\nu\rho}\frac{\partial x^\nu}{\partial\lambda}\frac{\partial x^\rho}{\partial\lambda} = 0$ which in this case gives two equations:

$$\frac{\partial^2 u(\lambda)}{\partial\lambda^2} - \Gamma^u_{uu}\frac{\partial x^u}{\partial\lambda}\frac{\partial x^u}{\partial\lambda} - 2\Gamma^u_{uv}\frac{\partial x^u}{\partial\lambda}\frac{\partial x^v}{\partial\lambda} - \Gamma^u_{vv}\frac{\partial x^v}{\partial\lambda}\frac{\partial x^v}{\partial\lambda} = \frac{\partial^2 u(\lambda)}{\partial\lambda^2} + \frac{1}{u}\frac{\partial x^u}{\partial\lambda}\frac{\partial x^u}{\partial\lambda} = 0$$
$$\frac{\partial^2 v(\lambda)}{\partial\lambda^2} - \Gamma^v_{uu}\frac{\partial x^u}{\partial\lambda}\frac{\partial x^u}{\partial\lambda} - 2\Gamma^v_{uv}\frac{\partial x^u}{\partial\lambda}\frac{\partial x^v}{\partial\lambda} - \Gamma^v_{vv}\frac{\partial x^v}{\partial\lambda}\frac{\partial x^v}{\partial\lambda} = \frac{\partial^2 v(\lambda)}{\partial\lambda^2} = 0$$

While $\{u(\lambda), v(\lambda)\} = \{\lambda, \lambda\}$ satisfies the second equation, it does not satisfy the first, so no this is not a geodesic.

3. Now consider the same set of points $\{u, v\}$, but now with metric $ds^2 = -v du^2 + dv^2$. You can choose to do either of the following questions, but do not need to do both in order to get full credit for this part. You grade will be based on the question that you get most correct.

   a) (*Mathematica*) Is this space maximally symmetric? Provide evidence.

      In Mathematica one can calculate the Ricci scalar (using the SCurvature command) and one finds that it depends on position, i.e. $R = \frac{1}{2v^2}$. This means that the space is not maximally symmetric since this critically depends on the Ricci scalar being a constant.

      I understand that some folks got the impression that satisfying $R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})$ alone is enough to prove that a space is maximally symmetric, however another, even more important criterion is that the Ricci scalar must be a constant. In this case this equation is satisfied, but the Ricci scalar is not constant. You will get credit for this line of reasoning since I wasn't perfectly clear in class.

   b) (*Mathematica*) What are the components of the energy-momentum tensor that would create this geometry?

      In Mathematica one can calculate the Einstein tensor (or the left hand side of Einstein's equation) divided by $8\pi G$ to find the energy-momentum tensor. In this case, the result is that all components are zero. This implies that the spacetime is Ricci flat, though not necessarily flat-flat (which it isn't).

   c) (*By hand*) Calculate by hand the Christoffel connection coefficients for this metric in these coordinates and then take the covariant derivative of the vector given by $A^\mu = (u, v)$.

$$\Gamma^v_{uu} = -\frac{1}{2}(g_{vv})^{-1}\partial_v g_{uu} = -\frac{1}{2}(1)^{-1}\partial_v(-v) = \frac{1}{2}, \qquad \Gamma^u_{vv} = -\frac{1}{2}(g_{uu})^{-1}\partial_u g_{vv} = 0$$

$$\Gamma^v_{uv} = \partial_u\left(ln\sqrt{|g_{vv}|}\right) = 0, \qquad \Gamma^u_{vu} = \partial_v\left(ln\sqrt{|g_{uu}|}\right) = \partial_v\left(ln\sqrt{v}\right) = \frac{1}{2v}$$

$$\Gamma^u_{uu} = \partial_u\left(ln\sqrt{|g_{uu}|}\right) = 0, \qquad \Gamma^v_{vv} = \partial_v\left(ln\sqrt{|g_{vv}|}\right) = 0$$

Then $\nabla_\nu A^\mu = \partial_\nu A^\mu + \Gamma^\mu_{\nu\lambda}A^\lambda$ which gives four equations:

$$\nabla_u A^u = \partial_u A^u + \Gamma^u_{uu}A^u + \Gamma^u_{uv}A^v = \frac{3}{2}$$

$$\nabla_u A^v = \partial_u A^v + \Gamma^v_{uu}A^u + \Gamma^v_{uv}A^v = \frac{1}{2}u$$

$$\nabla_v A^u = \partial_v A^u + \Gamma^u_{vu}A^u + \Gamma^u_{vv}A^v = \frac{1}{2v}u$$

$$\nabla_v A^v = \partial_v A^v + \Gamma^v_{vu}A^u + \Gamma^v_{vv}A^v = 1$$