Math Methods HW 4 Quiz Name___________________

You can try both problems below, but you will only receive credit for the most correct solution.

1. (*10 pts*) Consider the following matrix $M = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$. Show that it is normal. Then identify its further properties, i.e. Hermitian, unitary, symmetric, orthogonal. Find the eigenvalues of the matrix, then find the corresponding eigenvectors. Then find one of the matrices which diagonalize these, and identify the type of matrix it is, i.e. Hermitian, unitary, symmetric, orthogonal. And finally, using this matrix construct the diagonal form of the original matrix.

$MM^\dagger = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = M^\dagger M$ so M is normal. M is also Hermitian.

For the eigenvalues:

$\det[M - \lambda I] = 0 \;\Rightarrow\; (1-\lambda)^2 - 1 = 0 = \lambda^2 - 2\lambda + 1 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2) \;\Rightarrow\; \lambda = 0$ or 2.

For the eigenvectors:

$$\lambda = 0 \;\Rightarrow\; \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\Rightarrow\; \begin{matrix} a - ib = 0 \\ ia + b = 0 \end{matrix} \;\Rightarrow\; a = ib \;\Rightarrow\; \hat{x}_1 = \begin{pmatrix} i\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\lambda = 2 \;\Rightarrow\; \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix} \;\Rightarrow\; \begin{matrix} a - ib = 2a \\ ia + b = 2b \end{matrix} \;\Rightarrow\; a = -ib \;\Rightarrow\; \hat{x}_1 = \begin{pmatrix} -i\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$M_{diag} = P^{-1}MP$ with $P = \begin{pmatrix} i\frac{1}{\sqrt{2}} & -i\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ but P is unitary so $P^\dagger = P^{-1}$.

$$M_{diag} = \begin{pmatrix} -i\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ i\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} i\frac{1}{\sqrt{2}} & -i\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -i\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ i\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & -i\frac{2}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

2. (*10 pts*) Consider the matrix $M = \begin{pmatrix} 1+i & 0 \\ 0 & 1+i \end{pmatrix}$. Is this matrix Hermitian? Is it possible to transform this into an off-diagonal form via a similarity transformation? Explain why to both.

Is $M = M^\dagger$? $M^\dagger = \widetilde{M}^* = \begin{pmatrix} 1-i & 0 \\ 0 & 1-i \end{pmatrix}$, so no it is not Hermitian. Can we transform it to an off-diagonal form via a similarity transform? Well that will take $M \to M' = U^{-1}MU = U^{-1}(1+i)IU = (1+i)U^{-1}IU = (1+i)U^{-1}U = (1+i)I = M$ so the answer is no we cannot.