Math Methods HW 7 Quiz Name____________________

You can try both problems below, but you will only receive credit for the most correct solution.

1.(*10 pts*) Evaluate the Fourier transform of the product of the three "functions" $f_1(y) = \delta(y - y_1)$, $f_2(y) = \delta(y - y_2)$and $f_3(y) = \delta(y - y_3)$ where $y_1 \neq y_2 \neq y_3$. Verify that your answer makes sense.

First of all for $f_i(y) = \delta(y - y_i)$ we have $g_i(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(y - y_i)e^{-iky}dy = \frac{1}{\sqrt{2\pi}}e^{-iky_i}$.

Then

$$G(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(y - y_1)\delta(y - y_2)\delta(y - y_3)e^{-iky}dy$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-ik'y_1}\frac{1}{\sqrt{2\pi}}e^{-ik'''y_2}\frac{1}{\sqrt{2\pi}}e^{-i(k-k'-k''')y_3}dk'''dk'$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} e^{ik'(y_3-y_1)}\,dk'\int_{-\infty}^{\infty} e^{ik'''(y_3-y_2)}dk'''e^{-iky_3}$$

$$= \delta(y_3 - y_1)\delta(y_3 - y_2)e^{-iky_3} = 0 \text{ since } y_1 \neq y_2 \neq y_3.$$

We could have seen this straightaway from the starting point:

$$G(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(y - y_1)\delta(y - y_2)\delta(y - y_3)e^{-iky}dy = 0 \text{ since } y_1 \neq y_2 \neq y_3.$$

2. (*10 pts*) Show that $Y_{1-1} = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\varphi}$ using either Gram-Schmidt or Rodrigues. You may need to know that $\int_0^{\pi}\sin^3\theta\, d\theta = \frac{4}{3}$

To Gram-Schmidt it we start with:

$$\bar{Y}_{1-1} = Ae^{-i\varphi}\sin\theta$$

Then:

$$Y_{1-1} = \frac{\bar{Y}_{1-1} - Y_{00}(Y_{00},\bar{Y}_{1-1}) - Y_{10}(Y_{10},\bar{Y}_{1-1}) - Y_{11}(Y_{11},\bar{Y}_{1-1})}{\|\bar{Y}_{1-1} - Y_{00}(Y_{00},\bar{Y}_{1-1}) - Y_{10}(Y_{10},\bar{Y}_{1-1}) - Y_{11}(Y_{11},\bar{Y}_{1-1})\|}$$

So we need:

$$Y_{00}(Y_{00},\bar{Y}_{1-1}) = \frac{1}{2\sqrt{\pi}}\int_0^{2\pi}\int_0^{\pi}\frac{1}{2\sqrt{\pi}}Ae^{-i\varphi}\sin\theta\sin\theta\, d\theta d\varphi$$

$$= 0 \text{ since } \int_0^{2\pi} e^{-i\varphi}d\varphi = 0$$

$$Y_{10}(Y_{10},\bar{Y}_{1-1}) = \frac{\sqrt{3}}{2\sqrt{\pi}}\cos\theta\int_0^{2\pi}\int_0^{\pi}\frac{\sqrt{3}}{2\sqrt{\pi}}\cos\theta\, Ae^{-i\varphi}\sin\theta\sin\theta\, d\theta d\varphi$$

$$= 0 \text{ since } \int_0^{2\pi} e^{-i\varphi}d\varphi = 0$$

$$Y_{11}(Y_{11}, \bar{Y}_{1-1}) = -\frac{\sqrt{3}}{2\sqrt{2\pi}} e^{i\varphi} \sin\theta \int_0^{2\pi} \int_0^{\pi} -\frac{\sqrt{3}}{2\sqrt{2\pi}} e^{-i\varphi} \sin\theta\, Ae^{-i\varphi} \sin\theta\, d\theta d\varphi$$

$$= 0 \text{ since } \int_0^{2\pi} e^{-i2\varphi} d\varphi = 0$$

So in the end:

$$Y_{1-1} = \frac{\bar{Y}_{1-1}}{\|\bar{Y}_{1-1}\|} = \frac{Ae^{-i\varphi}\sin\theta}{\sqrt{\int_0^{2\pi}\int_0^{\pi} \sin\theta A e^{i\varphi} \sin\theta A e^{-i\varphi} \sin\theta\, d\varphi d\theta}} = \frac{e^{-i\varphi}\sin\theta}{\sqrt{\int_0^{2\pi}\int_0^{\pi} \sin^3\theta\, d\varphi d\theta}} = \frac{e^{-i\varphi}\sin\theta}{\sqrt{\frac{8\pi}{3}}} = \sqrt{\frac{3}{8\pi}} e^{-i\varphi} \sin\theta$$

Or we can use Rodrigues:

$$Y_{1-1}(\theta, \varphi) = (-1)^1 Y_{11}^* = (-1)(-1)^1 \left[\frac{3}{4\pi} \frac{0!}{2!}\right]^{\frac{1}{2}} P_1^1(\cos\theta) e^{-i\varphi}$$

Where

$$P_1^1(x) = (1-x^2)^{\frac{1}{2}} \frac{d^1}{dx^1} P_1(x) = (1-x^2)^{\frac{1}{2}} \frac{d^1}{dx^1} x = (1-x^2)^{\frac{1}{2}}$$

Then

$$Y_{32}(\theta, \varphi) = \sqrt{\frac{3}{8\pi}} \sqrt{1-\cos^2\theta}\, e^{-i\varphi} = \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\varphi}$$