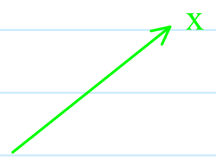
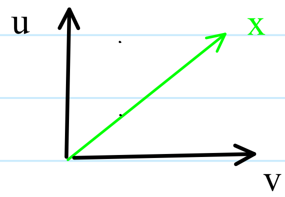
Matrices

Okay, so if any vector space V_n over F is isomorphic to F^n, then it is probably obvious that any linear operator on V_n is isomorphic to a matrix acting on elements of F^n via good old matrix multiplication!

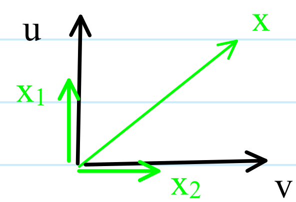
To get the notation aligned w/ the book consider a velocity vector V which obviously exists without the need for coordinates. However we can clean things up w/ coordinates.



Notice that w/ this you can still illustrate that it satisfies the defining properties of a vector space w/out coordinates. E.g. has an abelian group w/ +.



These are coordinates, but not yet a basis in the vector space. Not needed, but commonly used.



Now we have a coordinate adapted (unnecessary) basis. $X = \alpha_1 X_1 + \alpha_2 X_2 = \sum_{j=1}^{2} \alpha_j X_j$ but this is isomorphic to (α_1, α_2) a 2-tuple over $\mathbb{R}$ or $\mathbb{R}^2$

In principle, a linear transformation should be acting on the basis vectors (it is a transformation on vectors not scalars) which we write as $A x_j = \sum_{i=1}^{2} a_{ij} x_i$ which then gives $Ax = \sum_{j=1}^{2} \alpha_j \left(\sum_{i=1}^{2} a_{ij} x_i \right)$, that is the same components α_j but in a new basis x_j'. However we can easily interpret this as new components α_j' in terms of the old basis x_j by:

$$Ax = \sum_{i=1}^{2} \left(\sum_{j=1}^{2} a_{ij} \alpha_j \right) x_i$$ which represents the usual use of matrix multiplication!

E.g. $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\quad x = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ $\quad \ni \sum_j a_{ij} \alpha_j = \begin{pmatrix} a_{11}\alpha_1 + a_{12}\alpha_2 \\ a_{21}\alpha_1 + a_{22}\alpha_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$

As an example:

P_1 is the set of vectors of the form $x = \alpha_0 + \alpha_1 t$ and the linear transformation $D = \frac{d}{dt}$ carries these to $Dx = \alpha_1$. For a matrix realization consider

$$Dx = \begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix}\alpha_0 \\ \alpha_1\end{pmatrix} = \begin{pmatrix}\alpha_1 \\ 0\end{pmatrix} \Rightarrow D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$ using the basis $\{1, t\}$

Suppose instead we wanted to use the basis $\{1+t, 1-t\}$ then we have

$$x = \tfrac{1}{2}(\alpha_0 + \alpha_1)(1+t) + \tfrac{1}{2}(\alpha_0 - \alpha_1)(1-t) \Rightarrow x = \begin{pmatrix} \frac{1}{2}(\alpha_0 + \alpha_1) \\ \frac{1}{2}(\alpha_0 - \alpha_1) \end{pmatrix}$$

What about D?

Well $Dx = \alpha_1 = \frac{1}{2}\alpha_1(1+t) + \frac{1}{2}\alpha_1(1-t)$

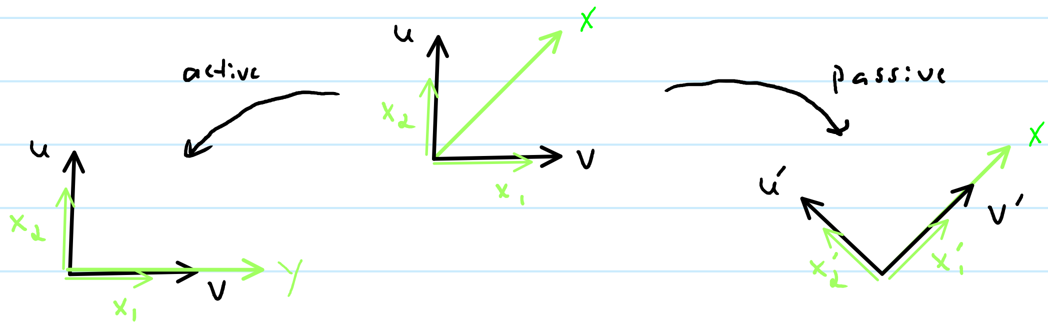
$$Dx = \begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} \frac{1}{2}(\alpha_0 + \alpha_1) \\ \frac{1}{2}(\alpha_0 - \alpha_1) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\alpha_1 \\ \frac{1}{2}\alpha_1 \end{pmatrix} \Rightarrow D = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

While it may be obvious that the sum of two transformations $A+B$ have matrix elements $a_{ij} + b_{ij}$, it may be less obvious that their product AB has $\sum_j a_{ij} b_{jk}$, which corresponds to ordinary matrix multiplication.

Other matrix features:

1) Diagonal matrices always commute.
2) Idempotent matrices satisfy $A^2 = A$ and include O, I.
3) Given the elements of A, a_{ij} then the transpose has elements $[\tilde{A}]_{ij} = a_{ji}$.
   Furthermore: if $\tilde{A} = B$ then $\tilde{B} = A$, $\widetilde{A+B} = \tilde{A} + \tilde{B}$ and $\widetilde{AB} = \tilde{B}\tilde{A}$
4) If A has complex elements then we can add $*$ to the transpose to form the adjoint w/ elements $[A^{\dagger}]_{ij} = a_{ji}^{*}$ and again if $A^{\dagger} = B \Rightarrow B^{\dagger} = A$, $(A+B)^{\dagger} = A^{\dagger} + B^{\dagger}$ and $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$
5) Symmetric $A \Rightarrow \tilde{A} = A$, Anti-symmetric $A \Rightarrow \tilde{A} = -A$ (w/ 0's on diag), any $A = \underbrace{\tfrac{1}{2}(A + \tilde{A})}_{\text{symm}} + \underbrace{\tfrac{1}{2}(A - \tilde{A})}_{\text{anti-symm}}$
6) A is nonsingular if there is a B s.t. $AB = I$, otherwise A is singular.

Now suppose we want to keep all vectors fixed, and just change the basis.
Therein we are led to the weirdness of "active" vs. "passive" transformations.



Note that "active" is how we define and described arbitrary linear transformations.
However "passive" is obviously related to basis/coordinate changes. In some cases these can be exchanged (rotations) but in many cases not $(A=0)$.
For arbitrary linear transformations A, we can have $y = Ax \neq x$, whereas if all we are doing is a basis/coordinate transformation, then $x' = \text{"}B\text{"}x = x$.
Now this might lead you to conclude that "B" $= I$ (which is not far off), but that is trying to interpret "B" as an active linear transformation.

So here we go: $x = \sum_i \alpha_i x_i = \sum_i \alpha_i' x_i' = \sum_i \underbrace{B^{-1}\alpha_i}_{\text{compensating component change}} \underbrace{B x_i}_{\text{changes basis}}$

more technically: if $B = \begin{pmatrix} b_{11} & b_{12} & \cdots \\ b_{21} & \cdots & \end{pmatrix}$

$x_i' = \sum_j b_{ji} x_j$ while $\underbrace{\alpha_i' = \sum_k (B^{-1})_{ik} \alpha_k}_{\text{don't be tempted by } b^{-1}_{ij}}$

as per book: $\alpha_i = \sum_j b_{ij} \alpha_j'$

Now this means that we can only work with B for which B^{-1} exists!
Going back to the active story, we may now take $\underset{\sum_i \beta_i x_i}{y} = \underset{A \sum_i \alpha_i x_i}{Ax}$, which all happens in x_i, and translate it into x_i'.

Specifically: $y = \sum_i \beta_i x_i = A \sum_i \alpha_i x_i = \sum_i \sum_j a_{ij} \alpha_j x_i$

becomes $y = \sum_i \beta_i' x_i' = A' \sum_i \alpha_i' x_i' = \sum_i \alpha_i' \sum_j a_{ji}' x_j' = \sum_i \sum_k (B^{-1})_{ik} \alpha_k \sum_j a_{ji}' \sum_\ell b_{\ell j} x_\ell$

$= \sum_\ell \sum_k \left(\sum_i \sum_j (B^{-1})_{ik} a_{ji}' b_{\ell j} \right) \alpha_k x_\ell \Rightarrow a_{\ell k} = \sum_i \sum_j (B^{-1})_{ik} a_{ji}' b_{\ell j}$

or rather $a_{ji}' = \sum_\ell \sum_k (B^{-1})_{j\ell} a_{\ell k} b_{ki} \Rightarrow A \to A' = B^{-1} A B$

So in the end we have: $y = A\alpha x \to y = A'\alpha' x' = B^{-1} A B B^{-1} \alpha B x = B^{-1} A \alpha B x = B^{-1}(A\alpha) B x$