Okay so normally a linear transformation (or isomorphic matrix) takes a vector to a different vector (or isomorphic n-tuple), but sometimes it gives back the same vector (in direction) albeit with a scalar multiple (which doesn't impact direction). The vector is called an eigenvector and the scalar an eigenvalue of the transformation.

Now let's collect what we are interested in. We consider a single linear transformation and then how it acts over an entire vector space. We are looking for special cases of this action.

We have: $A x_i = \lambda_i x_i$ (Note x_i is not necessarily a basis vector in this case, but if it satisfies this equation it is an eigenvector)

Now this is an expression involving a linear transformation of vectors, but recall we map $V \to$ n-tuple, then $A \to$ matrix. So henceforth we will work w/ matrices acting on components in a given basis.

Let's consider solving problems for a moment.

1. $Ax = y$ where A and y are known $\Rightarrow$ requires us to figure out A^{-1} then $x = A^{-1}y$
2. $Ax_i = \lambda_i x_i$ has both x_i and λ_i as unknowns. We need to work harder.

Consider: $Ax_i - \lambda_i x_i = 0 \stackrel{?}{=} (A - \lambda_i)x_i$ but wait, if A is a matrix and λ_i a scalar we need: $(A - I\lambda_i)x_i = 0 = Mx_i$ $M \equiv A - I\lambda_i$

Now imagine if M^{-1} existed, then $x_i = M^{-1}0 = 0$ and we're done.

To have an interesting ($\neq 0$) solution we need for M^{-1} not to exist.

The most robust predictor of whether M^{-1} exists is if $\det M = 0$ or not.

So for x_i to exist ($\neq 0$) we need $\det(A - I\lambda_i) = 0$ which only has λ_i unknowns. Of course this is a scalar equation, but of order n (for an $n \times n$ matrix) so we can expect up to n solutions (though not distinct), and we need to allow for complex ones since many (even totally real) polynomials need them, e.g. $x^2 + 1 = 0 \Rightarrow x = \pm i$.

Once we have a solution for λ_i, we plug it back into $Ax_i = \lambda_i x_i$ and solve for x_i.

An example:

θ is fixed and $\neq 0$ or π

Consider $V = \mathbb{R}^3$ and the rotation transformation $R_{xy} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

We can already determine one eigenvector, i.e. $X = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \Rightarrow R_{xy} X = X$ w/ eigenvalue +1

Let's find it with the tools.

First of all, $\det R_{xy} \neq 0$, but thats okay since we need $\det(R_x - I\lambda_i) = 0$.

Clearly if $\lambda_i = 1$ then this becomes $\det \begin{pmatrix} \cos\theta - 1 & -\sin\theta & 0 \\ \sin\theta & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which definitely $= 0$ (since an entire row and column $= 0$)

Let's treat λ_i as unknown:

Evaluating: $\det \begin{pmatrix} \cos\theta - \lambda_i & -\sin\theta & 0 \\ \sin\theta & \cos\theta - \lambda_i & 0 \\ 0 & 0 & 1-\lambda_i \end{pmatrix} = (\cos\theta - \lambda_i)(\cos\theta - \lambda_i)(1-\lambda_i) + \sin\theta\sin\theta(1-\lambda_i)$

$$= \lambda_i^3 - (2\cos\theta + 1)\lambda_i^2 + (2\cos\theta + 1)\lambda_i - 1$$

This equation has roots: $\lambda_i = \underset{\text{expected}}{1}, \cos\theta + i\sin\theta, \cos\theta - i\sin\theta = 1, \underbrace{e^{\pm i\theta}}_{\neq \pm 1 \text{ since } \theta \neq 0, \pi}$

Let's find the eigenvectors:

$\lambda_1 = 1 \quad R_{xy} X_1 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow a = b = 0 \quad c = \text{anything}$ (as expected)

$\lambda_2 = e^{i\theta} \quad R_{xy} X_2 = \begin{pmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \\ c \end{pmatrix} = \begin{pmatrix} e^{i\theta} a \\ e^{i\theta} b \\ e^{i\theta} c \end{pmatrix} \Rightarrow \left.\begin{aligned} a\cos\theta - b\sin\theta &= a\cos\theta + ai\sin\theta \\ a\sin\theta + b\cos\theta &= b\cos\theta + bi\sin\theta \\ c &= c\cos\theta + ci\sin\theta \end{aligned}\right\} \begin{aligned} -b &= ai \\ a &= bi \\ c &= 0 \end{aligned}$

$\lambda_3 = e^{-i\theta} \quad R_{xy} X_3 = \begin{pmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \\ c \end{pmatrix} = \begin{pmatrix} e^{-i\theta} a \\ e^{-i\theta} b \\ e^{-i\theta} c \end{pmatrix} \Rightarrow \left.\begin{aligned} a\cos\theta - b\sin\theta &= a\cos\theta - ai\sin\theta \\ a\sin\theta + b\cos\theta &= b\cos\theta - bi\sin\theta \\ c &= c\cos\theta - ci\sin\theta \end{aligned}\right\} \begin{aligned} b &= ai \\ \\ c &= 0 \end{aligned}$

So $X_1 = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$, $X_2 = \begin{pmatrix} a \\ -ia \\ 0 \end{pmatrix}$ and $X_3 = \begin{pmatrix} -ib \\ b \\ 0 \end{pmatrix}$ where a, b, c are anything $\neq 0$.

Now is this a story of "an $N \times N$ matrix has N eigenvalues, each of which provides one eigenvector, for a total of N eigenvalues and eigenvectors."
This is bullshit!

The algebraic multiplicity of an eigenvalue is the number of times it appears as a root to $\det(A - I\lambda) = 0$.

The geometric multiplicity of an eigenvalue is the number of linearly independent eigenvectors belonging to it.

The geometric multiplicity of an eigenvalue is less than or equal to the algebraic.

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \Rightarrow \det\begin{pmatrix} 1-\lambda & 0 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 = 0 \Rightarrow \lambda_1 = 1 \text{ and } \lambda_2 = 1 \quad \text{algebraic} = 2$$

$$A\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ -a+b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ b \end{pmatrix} \Rightarrow \text{geometric} = 1$$

Note: For a linear transformation, it always has at least one eigenvalue and eigenvector, but there is no guarantee that it will have enough linearly ind. eigenvectors to span the vector space on which it acts. Of course the largest number of linearly independent eigenvectors it can have is N.

If two matrices are similar, i.e. A is similar to B if $A = C^{-1}BC$, then they have the same eigenvalues w/ same geometric multiplicity.

To see part of the proof, consider $Bx_i = \lambda_i x_i$ then:

1. Apply C^{-1} to the left $C^{-1}Bx_i = \lambda_i C^{-1}x_i$ (C^{-1} commutes w/ scalar λ_i)
2. Insert $I = CC^{-1}$ in a special place $\underbrace{C^{-1}BC}C^{-1}x_i = \lambda_i C^{-1}x_i$

$$AC^{-1}x_i = \lambda_i C^{-1}x_i$$

$$Ay_i = \lambda_i y_i \quad \text{eigenvectors may differ but eigenvalues don't!}$$

Although the eigenvectors are not the same, if there are k linearly independent eigenvectors of B, then there also k of A.

This story simplifies for a diagonal matrix:

$$D = \begin{pmatrix} d_{11} & & \bigcirc \\ & d_{22} & \\ \bigcirc & & \ddots \; d_{NN} \end{pmatrix} \Rightarrow \det(D - \lambda I) = \prod_{i=1}^{N} (d_{ii} - \lambda) = 0 \Rightarrow \lambda_i = d_{ii}$$

$$\Downarrow$$

$$D x_i = \lambda_i x_i \Rightarrow x_i = \begin{pmatrix} 0 \\ \vdots \\ a_i \\ \vdots \\ 0 \end{pmatrix}$$

Combining this with the eigenvalues common to both a diagonal matrix D and any similar transformation A we have:

$D = P^{-1} A P$ where P is the matrix composed of the eigenvectors of A

Note that for the diagonal matrix we will always have N linearly independent eigenvectors. So too will A, and $\det P \neq 0$ as a result which of course means that P^{-1} exists.

If we start w/ A, then the only way that P exists is if A has a set of eigenvectors which span the space.

Back to our example: $A = R_{xy} \Rightarrow \lambda_1 = 1 \quad x_1 = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}, \quad \lambda_2 = e^{i\theta} \quad x_2 = \begin{pmatrix} a \\ -ia \\ 0 \end{pmatrix}, \quad \lambda_3 = e^{-i\theta} \quad x_3 = \begin{pmatrix} -ib \\ b \\ 0 \end{pmatrix}$

Then we have: $P = \begin{pmatrix} 0 & a & -ib \\ 0 & -ia & b \\ c & 0 & 0 \end{pmatrix} \Rightarrow P^{-1} R_{xy} P = \frac{1}{2abc} \begin{pmatrix} 0 & 0 & 2ab \\ bc & ibc & 0 \\ iac & ac & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & a & -ib \\ 0 & -ia & b \\ c & 0 & 0 \end{pmatrix}$

$$D = \begin{pmatrix} 1 & & \\ & e^{i\theta} & \\ & & e^{-i\theta} \end{pmatrix}$$

Note: $\det D = \prod_{i=1}^{N} \lambda_i$, but since $\det D = \det A$ for any similar $A \Rightarrow \det A = \prod_{i=1}^{N} \lambda_i$

On the other hand consider $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \Rightarrow A x_i = \lambda_i x_i = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \lambda \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} b + ac \\ c \end{pmatrix} = \begin{pmatrix} \lambda b \\ \lambda c \end{pmatrix}$

This only has the solution $x_1 = \begin{pmatrix} b \\ 0 \end{pmatrix}$ w/ $\lambda_1 = 1$ which do not span the space, hence A cannot be diagonalized by a similarity transformation.

Finally:

If $\det A = 0$ and the eigenvectors of A span V which A acts on, then $Ax = b$ has a solution if and only if b is a linear combination of eigenvectors for which $\lambda_i \neq 0$.

An example:

$$A = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 2 \end{pmatrix} \Rightarrow \det A = 0, \ \det(A - \lambda I) = 0 = (-\lambda)(1-\lambda)(2-\lambda) \Rightarrow \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2$$

$$Ax_1 = 0x_1 \Rightarrow x_1 = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \simeq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad Ax_2 = x_2 \Rightarrow x_2 = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} \simeq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad Ax_3 = 2x_3 \Rightarrow x_3 = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \simeq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Clearly these span V.

$$\underbrace{\begin{pmatrix} 0 & & \\ & 1 & \\ & & 2 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ v \\ 2w \end{pmatrix} = \begin{pmatrix} 0 \\ k \\ j \end{pmatrix}} \Rightarrow \underbrace{\begin{matrix} u = \text{anything} \\ v = k \\ 2w = j \end{matrix}}$$

Given $b = kx_2 + jx_3 \Rightarrow Ax = b$ $\qquad x = ux_1 + kx_2 + \frac{1}{2}jx_3$

but if $b = hx_1 + kx_2 + jx_3 \Rightarrow Ax = b$ has no solution (think about it!!)

$$\underbrace{\begin{pmatrix} 0 & & \\ & 1 & \\ & & 2 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ v \\ 2w \end{pmatrix} = \begin{pmatrix} h \\ k \\ j \end{pmatrix}} \quad \text{w/ } h \neq 0 \Rightarrow \text{no solution}$$