Recall the definition of adjoint matrices: A w/ elements $a_{ij} \longrightarrow A^{+}$ w/ elements $[A^{+}]_{ij} = a_{ji}^{*}$

or just $A^{+} = \tilde{A}^{*}$ s.t. $\begin{cases} A^{+} = B \Rightarrow A = B^{+} \text{ and } (A^{+})^{+} = A \\ (A+B)^{+} = A^{+} + B^{+} \\ (AB)^{+} = B^{+}A^{+} \end{cases}$ skip in lecture

Now let us instead define "adjoint" more abstractly in terms of linear operators.

Let A be a linear transformation on a vector space V.

For every A the operator A^{+} s.t. $(Ax, y) = (x, A^{+}y)$ for every $x, y \in V$ is called the adjoint.

or equivalently $(x, Ay) = (A^{+}x, y)$ since $(x, Ay) = \underbrace{(Ay, x)^{*}}_{(a,b)=(b,a)^{*}} = (y, A^{+}x)^{*} = (A^{+}x, y)$

Useful things about adjoints are that given A, A^{+} always exists and is unique, and A^{+} itself is a linear operator (all provable).

With this operator/inner-product definition we also find:

1. $(A+B)^{+} = A^{+} + B^{+}$
2. $(AB)^{+} = B^{+}A^{+}$
3. $(\alpha A)^{+} = \alpha^{*} A^{+}$ w/ α a scalar
4. $(A^{+})^{+} = A$

Let's prove (1) just to show that we do not need matrix properties to do so.

Start w/ $(x, C^{+}y) = (Cx, y)$ where C is a linear operator.

If $C = A + B$ then $(x, [A+B]^{+}y) = ([A+B]x, y) = (Ax + Bx, y)$ using linearity of operators

$= (Ax, y) + (Bx, y)$ using linearity of inner-product

$= (x, A^{+}y) + (x, B^{+}y)$ using definition of adjoint

$= (x, A^{+}y + B^{+}y)$ using linearity of inner-product

$= (x, [A^{+} + B^{+}]y)$ using linearity of operators

Now does this mean $[A+B]^{+} = A^{+} + B^{+}$? Well suppose we had $(x, Ay) = (x, By)$. Does this $\Rightarrow A = B$?

The answer seems to be yes, but suppose that x or y is 0. Then no!

In fact consider if A and B take y and project them to different subspaces which are orthogonal to x. Then no!

We can save it by saying, "if all of this holds for all values of x and y" because then we have the theorem if $(x, Ay) = 0$ for all x and $y \Rightarrow A = 0$, which applied to $(x, Ay) - (x, By) = 0 = (x, (A-B)y)$

$\Rightarrow A - B = 0 \Rightarrow A = B$

Similarly to prove (2):

$(x, [AB]^{+}y) = (ABx, y) = (Bx, A^{+}y) = (x, B^{+}A^{+}y)$ for all x and y (w/ theorem on $(x, Ay) = 0$)

Clearly, due to the isomorphism between matrices and linear operators, there should be a connection between the separate definitions of adjoint. There is!

[Let the matrix of A have components a_{ij} w.r.t. an orthonormal basis in X. Then the matrix $A^\dagger$ w.r.t. X is $[A^\dagger]_{ij} = a_{ji}^*$ (which was our matrix definition).

To prove the connection start with the matrix elements of a linear transformation w.r.t. the orthonormal basis $X = \{x_i\}$, i.e. $a_{ij} = \langle x_i, A x_j \rangle$. Then $a_{ij} = \underbrace{\langle A^\dagger x_i, x_j \rangle}_{\substack{\text{operator def.} \\ \text{of adjoint}}} = \underbrace{\langle x_j, A^\dagger x_i \rangle^*}_{\langle a,b\rangle = \langle b,a\rangle^*} = [A^\dagger]_{ji}^* \Rightarrow A^\dagger_{ij} = a_{ji}^*$

Now w/ the definition of adjoint in hand, we can specify a special class of linear operators.

[If $A = A^\dagger$, then A is "self-adjoint". $\begin{cases} \text{Real inner-product space} \Rightarrow \text{adjoint} = \text{symmetric } (A = \tilde{A}) \\ \text{Complex inner-product space} \Rightarrow \text{adjoint} = \text{Hermitian} \end{cases}$

I know you have worked w/ Hermitian operators/transformations in QM, and are familiar w/ some of their properties, e.g. real eigenvalues. So let's explore more mathematically.

[If A and B are self-adjoint $\Rightarrow$ so is $A+B$. (Obviously $[A+B]^\dagger = A^\dagger + B^\dagger = A + B = [A+B]$)

[If A is self-adjoint $\Rightarrow$ so is αA for real α. (Obviously $[\alpha A]^\dagger = \alpha^* A^\dagger = \alpha^* A = \alpha A$ if $\alpha = \alpha^*$)

[If A and B are self-adjoint $\Rightarrow AB$ is self-adjoint iff $[A,B] = 0$.

To prove the last one (which is less than obvious):

if : Assuming $AB = BA \Rightarrow (AB)^\dagger = B^\dagger A^\dagger = BA = AB$

only if : Assuming $(AB)^\dagger = AB \Rightarrow AB = (AB)^\dagger = B^\dagger A^\dagger = BA$

Recall the theorem:

(i) [A linear transformation A on an inner-product space is 0 if and only if $(x, Ay) = 0$ for all x, y.

It turns out that in certain situations this can be "strengthened".

(ii) [If A is a self-adjoint in a $\underbrace{\text{real inner-product space}}_{\text{Euclidean}}$, then $A = 0$ iff $\langle x, Ax \rangle = 0$ for all x,

and

(iii) [If A is any linear transformation in a $\overbrace{\text{complex inner-product space}}^{\text{unitary}}$, then $A = 0$ iff $\langle x, Ax \rangle = 0$ for all x.

Before proving them, let's consider their "strength." Strength is tied to the condition that must be met. For all x and y means that x and y may differ, whereas for all x in the latter two means you pair x to itself, but need not worry about when $x \neq y$.

(i, ii, iii) To prove necessity in all of these, note that if $A = 0 \Rightarrow \langle x, Ay \rangle = 0$ for all x and y including $x = y$.

To prove sufficiency, start w/ the first;

(i) If $\langle x, Ay \rangle = 0$ for all x and y then if we take $x = Ay$, $\langle Ay, Ay \rangle = 0 \Rightarrow Ay = 0$ for all y and therefore $A = 0$.

(ii) For the second, we'll show that the condition to be met actually reduces to that of the first.

If $\langle x, Ax \rangle = 0$ for all x then consider $\langle x+y, A[x+y] \rangle = 0 = \langle x, Ax \rangle + \langle x, Ay \rangle + \langle y, Ax \rangle + \langle y, Ay \rangle$

which gives $\underbrace{\langle x+y, A[x+y] \rangle}_{=0} - \underbrace{\langle x, Ax \rangle}_{=0} - \underbrace{\langle y, Ay \rangle}_{=0} = (x, Ay) + (y, Ax) = 0$ for all x and y.

but $(y, Ax) = (Ax, y)^* \overset{\text{since real}}{=} (Ax, y) = (x, A^+ y) \overset{\text{since self-adjoint}}{=} (x, Ay)$

then $2(x, Ay) = 0$ for all x, y but we have already shown $\Rightarrow A = 0$.

(iii) For the third we can actually use part of the proof of the second.

If $\langle x, Ax \rangle = 0$ for all x then following (ii) $(x, Ay) + (y, Ax) = 0$ for all x and y.

This time, since things can be imaginary, let's take $y \to iy$, then $(x, Aiy) + (iy, Ax) = 0$ for all x, y.

Then (recall $\langle \alpha x, \beta y \rangle = \alpha^* \beta (x, y)$) we have $i\langle x, Ay \rangle - i(y, Ax) = i[\underbrace{(x, Ay) - (y, Ax)}_{=0}] = 0$

Adding this to earlier we have $2(x, Ay) = 0$ for all x & y $\Rightarrow A = 0$.

Obviously we can combine the last two theorems into the statement:

[If A is self-adjoint (on real or complex V) then $A = 0$ iff $\langle x, Ax \rangle = 0$ for all x.

Now why all of this? Well vector spaces are used all over physics, but if you think about it, using a complex vector space seems to be less applicable since all physical quantities (things we measure) are real valued. But hopefully you are starting to see that some of the math is even more powerful when extended to a unitary space. For example theorem (ii) works for self-adjoints, while theorem (iii) works for any.

So is there a way out of needing the advantages of complexity, but being restricted to real measurables? The answer is yes! And it relies on equating physical measurables to Hermitian operators/transformations.

Here's two wonderful results:

[A linear transformation A, on a unitary space, is Hermitian if and only if (x, Ax) is real for all x.

This let's us safely (and usefully) define a measured quantity w.r.t. (x, Ax). What is it? It's the expectation value of A w.r.t. the state identified by x. That is, it is the average value of A obtained over many measurements (so long as x is normalized).

Well that's the average value of A. What about the possible results on single measurements? Well...

[The eigenvalues of a Hermitian operator/transformation are real.

These are them!!

These play such an important role for opening the connection between complex vector spaces and complex operators to real quantities which is the mathematical backbone of QM. Let's prove.

For the first:

using

if: If $A = A^+$ then $(x, Ax) = (A^+x, x) = (Ax, x) = (x, Ax)^*$ $\Rightarrow$ (x, Ax) is real since $(\,)=(\,)^*$

and only if: If $(x, Ax) = (x, Ax)^* = (Ax, x) = (x, A^+x)$ (note we haven't used $A = A^+$)

Then $(x, [A - A^+]x) = 0$ for all x $\Rightarrow$ $A - A^+ = 0$ $\Rightarrow$ $A = A^+$

And the second:

$Ax = \lambda x \Rightarrow (x, Ax) = (x, \lambda x) = \lambda \|x\|^2 \Rightarrow \lambda = \frac{(x, Ax)}{\|x\|^2}$ but (x, Ax) is real by the previous as is $\|x\|^2$.

Another subgroup of linear transformations which is extremely useful are called "isometries."

For a linear transformation U on an inner-product space V we define:

If V is complex and $U^+U = UU^+ = I$ then U is "unitary"

If V is real and $\tilde{U}U = U\tilde{U} = I$ then U is "orthogonal"

} In both cases U is an isometry

Note these imply: $(U^+)^+ = U = (U^{-1})^+ = (U^+)^{-1} = (U^{-1})^{-1}$ and similarly for $\sim$

Now this definition doesn't really use the inner-product, but it turns out that these transformations are particularly important for inner-product spaces. In the following, if one is true, all are true.

U on an inner-product space satisfies:

1. $U^+U = I$
2. $(Ux, Uy) = (x, y)$ for all x and y
3. $\|Ux\| = \|x\|$ for all x

To prove it, start w/ 1 being true: $U^+U = I \Rightarrow (Ux, Uy) = (x, U^+Uy) = (x, y)$ for all x, y

and if $x = y$ $\langle Ux, Ux \rangle = \|Ux\|^2 = (x, x) = \|x\|^2$ for all x

So $1 \Rightarrow 2 \Rightarrow 3$, and to finish up we need $3 \Rightarrow 1$

If $\|Ux\|^2 = \langle Ux, Ux \rangle = \underbrace{(U^+Ux, x)}_{\text{we're not saying } U^+U = I \text{ yet!}} = (x, x) \Rightarrow ([U^+U - I]x, x) = 0$ for all x

This is just another expression for what is on the left of it.

Recall that if $(Ax, x) = 0$ for all x and A is self-adjoint then $A = 0$.

So if $[U^+U - I]$ is self adjoint then $U^+U - I = 0 \Rightarrow U^+U = I$ and 1 holds.

Is $[U^+U - I]$ self-adjoint? $[U^+U - I]^+ = (U^+U)^+ - I^+ = U^+U^{++} - I = U^+U - I$

so yes it is!

So isometries preserve lengths of vectors, but they also preserve the angle between two vectors

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\|\|y\|} \Rightarrow \frac{\langle Ux, Uy \rangle}{\|Ux\|\|Uy\|} = \frac{\langle x, y \rangle}{\|x\|\|y\|} = \cos\theta$$

But the preservation of lengths and angles means that an isometry will carry one $\underbrace{\text{ortho}}_{\text{angle}=90^\circ}\underbrace{\text{normal}}_{\text{length}=1}$ set into another. The only concern might be whether it takes a complete orthonormal basis into another, and this can be proven w/ Parseval's equation.

So we have: [If $\{x_i\}$ is a complete orthonormal basis, then so is $\{Ux_i\}$ for any isometry U.

Going back to eigenvalues, we can draw the results for self-adjoint and isometric transformations:

For Hermitian transformations we know the eigenvalues are all real. But the same is true for symmetric transformations. Which leads to the slightly more general:

[If A is a self-adjoint transformation, then all of its eigenvalues are real.

To prove this recall eigenvalues λ of A are s.t. $Ax = \lambda x$ for $x \neq 0$.

Now $\langle x, Ax \rangle = \langle x, \lambda x \rangle = \lambda \langle x, x \rangle$, but if $A = A^+$ then $\langle x, Ax \rangle = \langle A^+ x, x \rangle = \langle Ax, x \rangle = \lambda^* \langle x, x \rangle$

so we have $\lambda \langle x, x \rangle = \lambda^* \langle x, x \rangle$ and $\langle x, x \rangle \neq 0$ so $\lambda = \lambda^*$.

For isometric transformations we have:

[All eigenvalues of isometric transformations have absolute values of 1.

Proof:

If U is an isometry and $Ux = \lambda x$ for $x \neq 0$ then $\|x\| = \|Ux\| = |\lambda| \|x\| \Rightarrow |\lambda| = 1$

since $\|x\| \neq 0$.