Examples: Symmetric $A = \tilde{A}$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow \lambda_1 = -1 \quad \hat{x}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\lambda_2 = 3 \quad \hat{x}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

and $A_{diag} = P^{-1} A P$ w/ $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

or

$P' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

both of which are orthogonal $\tilde{P} P = I$

$$A_{diag} = P^{-1} A P = \tilde{P} A P = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$A_{diag} = P'^{-1} A P' = \tilde{P}' A P' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

Antisymmetric and orthogonal $B = -\tilde{B}$ $\quad B\tilde{B} = I$

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \lambda_1 = i \quad \hat{x}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ i\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\lambda_2 = -i \quad \hat{x}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -i\frac{1}{\sqrt{2}} \end{pmatrix}$$

and $B_{diag} = P^{-1} B P$ w/ $P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -i\frac{1}{\sqrt{2}} & i\frac{1}{\sqrt{2}} \end{pmatrix}$

or

$P' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ i\frac{1}{\sqrt{2}} & -i\frac{1}{\sqrt{2}} \end{pmatrix}$

both of which are unitary $P^{+} P = I$

$$B_{diag} = P^{-1} B P = P^{+} B P = \begin{pmatrix} \frac{1}{\sqrt{2}} & i\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -i\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -i\frac{1}{\sqrt{2}} & i\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & i\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -i\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -i\frac{1}{\sqrt{2}} & i\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$B_{diag} = P'^{-1} B P' = P'^{+} B P' = \begin{pmatrix} \frac{1}{\sqrt{2}} & -i\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & i\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ i\frac{1}{\sqrt{2}} & -i\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -i\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & i\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} i\frac{1}{\sqrt{2}} & -i\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

What more does adjointing (but not self-adjoint) bring us?

[ If $A$ is any linear operator w/ eigenvalues $\lambda_i$, then $A^+$ has eigenvalues $\lambda_i^*$.

Proof: Recall $\lambda$ come from $\det(A-\lambda I)=0$, but $\det A = (\det A^+)^* \Rightarrow (\det A)^* = \det A^+$

and since $\det(A-\lambda I) = 0 = [\det(A-\lambda I)]^* \Rightarrow \det(A-\lambda I)^+ = \underbrace{0 = \det(A^+ - \lambda^* I)}_{\text{hence } \lambda^* \text{ are eigenvalues of } A^+}$

Moreover:

[ If any $A$ has an eigenvalue $\lambda_i$ w/ an eigenvector $x_i$, and $A^+$ has an eigenvalue $\lambda_j^*$ w/ eigenvector $x_j$, then $\langle x_i, x_j \rangle = 0$ whenever $\lambda_i \neq \lambda_j$.

Proof: Consider $Ax_i = \lambda_i x_i \Rightarrow \langle x_j, Ax_i \rangle = \lambda_i \langle x_j, x_i \rangle$

def. of adjoint

$\langle A^+ x_j, x_i \rangle = \langle \lambda_j^* x_j, x_i \rangle = \lambda_j \langle x_j, x_i \rangle$

$\Rightarrow (\lambda_i - \lambda_j)\underbrace{\langle x_j, x_i \rangle}_{=0 \text{ if } \lambda_i \neq \lambda_j} = 0$

So now we are going to flash through some results from the sections on "The Solvability of Linear Equations" and "Minimum Principles." No proof will be provided (they are in the book), but rather we will illustrate these by example.

[ If $A$ is a linear operator on a vector space w/ eigenvalues $\lambda_i$, then $A^\dagger$ has eigenvalues $\lambda_i^*$.

Consider $A = \begin{pmatrix} i+1 & 0 & 0 \\ 0 & i & 1 \\ 0 & 1 & i \end{pmatrix} \Rightarrow \det\begin{pmatrix} i+1-\lambda & 0 & 0 \\ 0 & i-\lambda & 1 \\ 0 & 1 & i-\lambda \end{pmatrix} = 0 = (i+1-\lambda)\left[(i-\lambda)^2-1\right] = (i+1-\lambda)(\lambda^2-2i\lambda-2)$

then $\lambda_1 = i+1$, $\lambda_2 = \frac{2i+\sqrt{-4+8}}{2} = i+1$, $\lambda_3 = \frac{2i-\sqrt{-4+8}}{2} = i-1$

Consider $A^* = \begin{pmatrix} -i+1 & 0 & 0 \\ 0 & -i & 1 \\ 0 & 1 & -i \end{pmatrix} \Rightarrow \det\begin{pmatrix} -i+1-\lambda^* & 0 & 0 \\ 0 & -i-\lambda^* & 1 \\ 0 & 1 & -i-\lambda^* \end{pmatrix} = 0 = (-i+1-\lambda^*)(\lambda^{*2}+2i\lambda^*-2)$

then $\lambda_1^* = -i+1$, $\lambda_2 = -i+1$, $\lambda_3 = -i-1$ (As promised by theorem)

[ Consider $A$ w/ $\lambda$ and $A^\dagger$ w/ $\lambda^*$. The geometric multiplicity of $\lambda$ is the same as $\lambda^*$.

[ If $\{\lambda_i\}$ is the spectrum of $A$ (then $\{\lambda_i^*\}$ is that of $A^\dagger$) then if $\phi_j$ is an $\overbrace{\text{eigenfunction}}^{\text{eigenvector}}$ of $A$ belonging to $\lambda_j$, and $\psi_i$ is an eigenfunction of $A^\dagger$ belonging to $\lambda_i^*$, then $(\phi_j, \psi_i) = 0$ if $\lambda_i \neq \lambda_j$.

Eigenvectors of $A$: $\lambda_{1,2} \Rightarrow \begin{pmatrix} i+1 & 0 & 0 \\ 0 & i & 1 \\ 0 & 1 & i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ia+a \\ ib+b \\ ic+c \end{pmatrix} \Rightarrow \begin{matrix} ia+a = ia+a \\ ib+c = ib+b \\ b+ic = ic+c \end{matrix} \Rightarrow \begin{matrix} a = \text{whatever} \\ b = c \end{matrix} \Rightarrow \hat{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \overset{\text{G.S.}}{\Rightarrow} \hat{x}_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$\lambda_3 \Rightarrow \begin{pmatrix} i+1 & 0 & 0 \\ 0 & i & 1 \\ 0 & 1 & i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ia-a \\ ib-b \\ ic-c \end{pmatrix} \Rightarrow \begin{matrix} ia+a = ia-a \\ ib+c = ib-b \\ b+ic = ic-c \end{matrix} \Rightarrow \begin{matrix} a = 0 \\ b = -c \end{matrix} \Rightarrow \hat{x}_3 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

Eigenvectors of $A$: $\lambda_{1,2}^* \Rightarrow \begin{pmatrix} -i+1 & 0 & 0 \\ 0 & -i & 1 \\ 0 & 1 & -i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -ia+a \\ -ib+b \\ -ic+c \end{pmatrix} \Rightarrow \begin{matrix} -ia+a = -ia+a \\ -ib+c = -ib+b \\ b-ic = -ic+c \end{matrix} \Rightarrow \begin{matrix} a = \text{whatever} \\ b = c \end{matrix} \Rightarrow \hat{y}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \overset{\text{G.S.}}{\Rightarrow} \hat{y}_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$\lambda_3^* \Rightarrow \begin{pmatrix} -i+1 & 0 & 0 \\ 0 & -i & 1 \\ 0 & 1 & -i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -ia-a \\ -ib-b \\ -ic-c \end{pmatrix} \Rightarrow \begin{matrix} -ia+a = -ia-a \\ -ib+c = -ib-b \\ b-ic = -ic-c \end{matrix} \Rightarrow \begin{matrix} a = 0 \\ b = -c \end{matrix} \Rightarrow \hat{y}_3 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

Note: $\underbrace{(\hat{x}_1, \hat{y}_3)}_{\lambda_1 \neq \lambda_3} = \underbrace{(\hat{x}_2, \hat{y}_3)}_{\lambda_2 \neq \lambda_3} = \underbrace{(\hat{x}_3, \hat{y}_1)}_{\lambda_3 \neq \lambda_1} = \underbrace{(\hat{x}_3, \hat{y}_2)}_{\lambda_3 \neq \lambda_2} = 0$

And finally we have a generalization of:

If $\det A = 0$ and the eigenvectors of $A$ span $V$, then $Ax = b$ has a solution iff $b$ is a lin. comb. of eigenvectors for which $\lambda_i \neq 0$.

to

The equation $(A - \lambda I)x = y$ has a solution iff $(\phi_i, y) = 0$ for all $\phi_i$ satisfying $(A^{+} - \lambda^{*} I)\phi_i = 0$.

For example take $\lambda_1 = i+1 \Rightarrow$ $$\underbrace{\begin{pmatrix} i+1-i-1 & 0 & 0 \\ 0 & i-i-1 & 1 \\ 0 & 1 & i-i-1 \end{pmatrix}}_{A - \lambda I} \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ -b+c \\ b-c \end{pmatrix} = \underbrace{\begin{pmatrix} d \\ e \\ f \end{pmatrix}}_{y}$$

and $$\underbrace{\begin{pmatrix} -i+1+i-1 & 0 & 0 \\ 0 & -i+i-1 & 1 \\ 0 & 1 & -i+i-1 \end{pmatrix}}_{A^{+} - \lambda^{*} I} \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{\phi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \phi = \begin{pmatrix} a \\ b \\ b \end{pmatrix} \Rightarrow \phi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \overset{G.S.}{\Rightarrow} \phi_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Let $\underbrace{y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{(y, \phi_1) \neq 0} \Rightarrow \left.\begin{matrix} 0 = 1 \\ -b+c = 0 \\ b-c = 0 \end{matrix}\right\}$ no solution, $\underbrace{y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{(y, \phi_2) \neq 0} \Rightarrow \left.\begin{matrix} 0 = 0 \\ -b+c = 1 \\ b-c = 0 \end{matrix}\right\}$ no solution, $\underbrace{y = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{(y, \phi_2) \neq 0} \Rightarrow \left.\begin{matrix} 0 = 0 \\ -b+c = 0 \\ b-c = 1 \end{matrix}\right\}$ no solution

But $\underbrace{y = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{(y, \phi_1) = 0 = (y, \phi_2)} \Rightarrow \left.\begin{matrix} 0 = 0 \\ -b+c = 1 \\ b-c = -1 \end{matrix}\right\} \begin{pmatrix} a \\ b \\ 1+b \end{pmatrix}$ is a solution!

From "Minimum Principles":

$\Big[$ If $A$ is self-adjoint, $Ax = \lambda x$ if and only if $I = (x, Ax)$ is extremized w.r.t. the constraint $(x,x) = 1$. The extremized value of $I$ is just $\lambda$.

Recall: $A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$ first of all $A^{+} = A$

The eigenvalues are $\lambda_i = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{3, 5, -1, -1\}$

The eigenvectors are

$$X_1 = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad a = \text{anything} \Rightarrow \hat{X}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 0 \\ b \\ b \\ b \end{pmatrix} \quad b = \text{anything} \Rightarrow \hat{X}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$X_3 = \begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix} \quad a + b + c = 0$$

$$X_4 = \begin{pmatrix} 0 \\ d \\ e \\ f \end{pmatrix} \quad d + e + f = 0$$

Let's see this in the extremising language:

$$I = (a\ b\ c\ d)\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 3a^2 + b^2 + c^2 + d^2 + 4bc + 4cd + 4bd$$

subject to $(x,x) = a^2 + b^2 + c^2 + d^2 = 1$ or $J = a^2 + b^2 + c^2 + d^2 - 1 = 0$

Lagrange multiplier

To extremize consider $K = I - \lambda J = (3-\lambda)a^2 + (1-\lambda)b^2 + (1-\lambda)c^2 + (1-\lambda)d^2 + \lambda + 4bc + 4bd + 4cd$

Now demand $\frac{\partial K}{\partial a} = \frac{\partial K}{\partial b} = \frac{\partial K}{\partial c} = \frac{\partial K}{\partial d} = 0$

$\frac{\partial K}{\partial a} = (6 - 2\lambda)a = 0 \qquad \frac{\partial K}{\partial c} = (1-\lambda)2c + 4b + 4d = 0$

$\frac{\partial K}{\partial b} = (1-\lambda)2b + 4c + 4d = 0 \qquad \frac{\partial K}{\partial d} = (1-\lambda)2d + 4b + 4c = 0$

Turn the $\lambda$ dial:

$\lambda = 3 \Rightarrow (0)a = 0 \Rightarrow a = \text{anything}$

$$\left.\begin{aligned} -4b + 4c + 4d &= 0 \\ -4c + 4b + 4d &= 0 \\ -4d + 4b + 4c &= 0 \end{aligned}\right\} d = b = c = 0$$

$(x,x) = 1 \Rightarrow a = 1 \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$I = 3a^2 = 3$

$\lambda = 5 \Rightarrow -4a = 0 \Rightarrow a = 0$

$$\left.\begin{aligned} -8b + 4c + 4d &= 0 \\ -8c + 4b + 4d &= 0 \\ -8d + 4b + 4c &= 0 \end{aligned}\right\} b = c = d$$

$(x,x) = 1 \Rightarrow b = c = d = \frac{1}{\sqrt{3}} \Rightarrow \begin{pmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

$I = 5(b^2 + c^2 + d^2) = 5$

$\lambda = -1 \Rightarrow 4a = 0$

$$\left.\begin{aligned} 4b + 4c + 4d &= 0 \\ 4c + 4b + 4d &= 0 \\ 4d + 4b + 4c &= 0 \end{aligned}\right\} b + c + d = 0$$

$(x,x) = 1 \Rightarrow \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ -2/\sqrt{3} \end{pmatrix}$

$I = \underbrace{(b+c+d)^2}_{=0} + 2bc + 2cd + 2bd = -1$

For self-adjoint $A$, if $\lambda$ is smallest eigenvalue and $\lambda'$ the largest, then the max/min of $(x, Ax)$ are $\lambda'/\lambda$ for all vectors s.t. $(x, x) = 1$.

Now consider:

If $V$ is a vector space and $\mathcal{U}$ is a subspace of $V$, the set of vectors not in $\mathcal{U}$ is the "complement" of $\mathcal{U}$ in $V$, $\mathcal{U}_c$.

Then:

If $A$ is self-adjoint in an $n$-dim. $V$ w/ eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, consider $\mathcal{U}_i$: the subspace of $V$ spanned by eigenvectors associated w/ $\lambda$ up to $\lambda_i$. Then normalized vectors $\{x\}$ that are in the complement $\mathcal{U}_{ic}$, the minimum of $(x, Ax)$ is assumed when $x = x_{i+1}$ w/ $\lambda_{i+1}$.

Consider vectors in $\mathcal{U}_{ic}$ where $\mathcal{U}_i$ is spanned by $\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ -2/\sqrt{3} \end{pmatrix}$

i.e. $x \in \mathcal{U}_{ic}$ when $x = a\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} a \\ b/\sqrt{3} \\ b/\sqrt{3} \\ b/\sqrt{3} \end{pmatrix}$

Then $K = I - \lambda J = 3a^2 + 5b^2 - \lambda a^2 - \lambda b^2 + \lambda$

$$= (3-\lambda)a^2 + (5-\lambda)b^2 + \lambda$$

$$\left.\begin{aligned} \frac{\partial K}{\partial a} &= (6 - 2\lambda)a = 0 \\ \frac{\partial K}{\partial b} &= (10 - 2\lambda)b = 0 \end{aligned}\right\} \begin{aligned} &\Rightarrow \lambda = 3 \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &\Rightarrow \lambda = 5 \quad \begin{pmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \end{aligned}$$

$\Downarrow$

$\lambda = -1 \Rightarrow$ no solution