Perturbation Theory

A simple story:

Suppose you were asked to find $\sin(91^\circ)$ w/out a calculator.

Well you know that $\sin(90^\circ) = 1$, so why not call Taylor.

$$f(x_0+\epsilon) = f(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(x_0)\, \epsilon^n$$

$$\sin(91^\circ) = \sin(90^\circ) + \cos(90^\circ)\text{"}1^\circ\text{"} - \frac{1}{2}\sin(90^\circ)\text{"}1^\circ\text{"}^2 - \dots$$

where "1°" $= 0.01745\dots$ i.e. "1°" $\ll 1$ so we can keep just the first few terms.

We can apply similar reasoning to the Eigen problem, given A find λ and x s.t. $Ax = \lambda x$.

Before getting started.

So far we know:

For a normal linear transformation A, i.e. $[A, A^+] = 0$, then A always has n-eigenvalues and n-eigenvectors, and the algebraic multiplicity of an eigenvalue matches the geometric multiplicity of the associated eigenvectors (which it must based on the previous statement). Furthermore, the eigenvectors associated w/ distinct eigenvalues are orthogonal, while those associated w/ degenerate eigenvalues are linearly independent w.r.t. to all other eigenvectors, and orthogonal w.r.t. to those associated with other eigenvalues.

Nondegenerate Perturbation Theory

It is easy to imagine the task of finding eigenvalues and eigenvectors for a linear operator getting hard. But hopefully, now you see the value in doing it (makes solving systems of equations much easier when using the diagonal form). So let's consider self-adjoint operators and explore an approximation scheme to solving this problem.

self-adjoint

Consider self-adjoint $A = A_0 + \epsilon A_1 = A^+$ w/ A_0 something we know about (eigens) and ϵ small.
Imagine having solutions to $A_0 x^{(0)} = \lambda^{(0)} x^{(0)}$ (we will start w/ a single eigenvalue λ and eigenvector x)
We will seek solutions to $Ax = \lambda x$ (again a single eigenvalue λ and eigenvector x) of the form:

$$\lambda = \lambda^{(0)} + \epsilon \lambda^{(1)} + \cdots = \sum_{i=0}^{\infty} \epsilon^i \lambda^{(i)} \qquad (i) \text{ is a label}$$

$$x = x^{(0)} + \epsilon x^{(1)} + \cdots = \sum_{i=0}^{\infty} \epsilon^i x^{(i)} \qquad i \text{ is a power}$$

Plugging these in: $$Ax = \underbrace{(A_0 + \epsilon A_1) \sum_{i=0}^{\infty} \epsilon^i x^{(i)}}_{\text{l.h.s.}} = \lambda x = \underbrace{\sum_{i=0}^{\infty} \epsilon^i \lambda^{(i)} \sum_{j=0}^{\infty} \epsilon^j x^{(j)}}_{\text{r.h.s.}}$$

We would like to peel off terms w/ the same power of ϵ. To do so requires some rewriting.

l.h.s. $$(A_0 + \epsilon A_1)\sum_{i=0}^{\infty} \epsilon^i x^{(i)} = \sum_{i=0}^{\infty} \epsilon^i A_0 x^{(i)} + \sum_{i=0}^{\infty} \epsilon^{i+1} A_1 x^{(i)} = \sum_{i=0}^{\infty} \epsilon^i A_0 x^{(i)} + \sum_{i=0}^{\infty} \epsilon^i A_1 \underbrace{x^{(i-1)}}_{\text{if } x^{-1}=0}$$

r.h.s. $$\sum_{i=0}^{\infty} \epsilon^i \lambda^{(i)} \sum_{j=0}^{\infty} \epsilon^j x^{(j)} = \underbrace{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \epsilon^{i+j} \lambda^{(i)} x^{(j)}}_{\underbrace{\epsilon^0 \lambda^{(0)} x^{(0)}}_{i=j=0} + \underbrace{\epsilon^1 \lambda^{(1)} x^{(0)}}_{i=1,\ j=0} + \underbrace{\epsilon^1 \lambda^{(0)} x^{(1)}}_{i=0\ j=1} + \cdots} = \underbrace{\sum_{i=0}^{\infty}\sum_{j=0}^{i} \epsilon^i \lambda^{(j)} x^{(i-j)}}_{\underbrace{\epsilon^0 \lambda^{(0)} x^{(0)}}_{i=0} + \underbrace{\epsilon^1 \lambda^{(0)} x^{(1)} + \epsilon \lambda^{(1)} x^{(0)}}_{i=1}}$$

Now we have: $$\sum_{i=0}^{\infty} \epsilon^i \left[A_0 x^{(i)} + A_1 x^{(i-1)} \right] = \sum_{i=0}^{\infty} \epsilon^i \left[\sum_{j=0}^{i} \lambda^{(j)} x^{(i-j)} \right].$$

So for each power of ϵ^i: $$A_0 x^{(i)} + A_1 x^{(i-1)} = \sum_{j=0}^{i} \lambda^{(j)} x^{(i-j)} = \lambda^{(0)} x^{(i)} + \sum_{j=1}^{i} \lambda^{(j)} x^{(i-j)}$$

Rearranging: $$\left[A_0 - \lambda^{(0)}\right] x^{(i)} = \sum_{j=1}^{i} \lambda^{(j)} x^{(i-j)} - A_1 x^{(i-1)} \qquad i = 0, 1, \cdots$$

$i=0$ $\left[A_0 - \lambda^{(0)}\right] x^{(0)} = 0$ We know everything in this starting point.

$i=1$ $\left[A_0 - \lambda^{(0)}\right] x^{(1)} = \lambda^{(1)} x^{(0)} - A_1 x^{(0)} = -\left[A_1 - \lambda^{(1)}\right] x^{(0)}$

$i=2$ $\left[A_0 - \lambda^{(0)}\right] x^{(2)} = \lambda^{(1)} x^{(1)} + \lambda^{(2)} x^{(0)} - A_1 x^{(1)} = -\left[A_1 - \lambda^{(1)}\right] x^{(1)} + \lambda^{(2)} x^{(0)}$

$i=3$ $\left[A_0 - \lambda^{(0)}\right] x^{(3)} = \lambda^{(1)} x^{(2)} + \lambda^{(2)} x^{(1)} + \lambda^{(3)} x^{(0)} - A_1 x^{(2)} = -\left[A_1 - \lambda^{(1)}\right] x^{(2)} + \lambda^{(2)} x^{(1)} + \lambda^{(3)} x^{(0)}$

If we solve $0 \to 1 \to 2 \to \cdots$ note that each new equation brings in 2 new unknowns $\lambda^{(i)}$ and $x^{(i)}$, but...

We could do some trickery and obtain expressions for $\lambda^{(i)}$ in terms of $\lambda^{(j)}$ and $x^{(j)}$ w/ $j<i$, but instead we will focus on the existence of solutions (and use that to do it).

Recall the theorem from the last lecture:

[$(A-\lambda I)x = y$ has a solution iff $\langle \psi_i, y \rangle = 0$ for ψ_i satisfying $(A^{+}-\lambda^{*} I)\psi_i = 0$

Well this will obviously be useful for our equations as they are written.

In these we have: $(A_0 - \lambda^{(0)} I)\, x^{(i)} = y$

Note that since A_0 is self adjoint $A_0^{+} = A_0$ and $\lambda^{(0)*} = \lambda^{(0)}$ so we need ψ_i which satisfies $(A_0 - \lambda^{(0)} I)\psi_i = 0 \Rightarrow \psi_i = x^{(0)}$!

Now for a solution to exist we just need $\langle x^{(0)}, y \rangle = 0$ in each equation.

So we will have a solution to the first equation if:

$\langle x^{(0)}, -[A_1 - \lambda^{(1)}]x^{(0)} \rangle = 0 \Rightarrow (x^{(0)}, \lambda^{(1)} x^{(0)}) = (x^{(0)}, A_1 x^{(0)}) \Rightarrow \boxed{\lambda^{(1)} = \langle x^{(0)}, A_1 x^{(0)} \rangle}$

To the second if:

$\langle x^{(0)}, -[A_1 - \lambda^{(1)}]x^{(1)} + \lambda^{(2)} x^{(0)} \rangle = 0 \Rightarrow \lambda^{(2)} = \langle x^{(0)}, [A_1 - \lambda^{(1)}]x^{(1)} \rangle$

And the third:

$\langle x^{(0)}, -[A_1 - \lambda^{(1)}]x^{(2)} + \lambda^{(2)} x^{(1)} + \lambda^{(3)} x^{(0)} \rangle = 0 \Rightarrow \lambda^{(3)} = \langle x^{(0)}, [A_1 - \lambda^{(1)}]x^{(2)} \rangle - \lambda^{(2)} \langle x^{(0)}, x^{(1)} \rangle$

It turns out that w/ some more work we can simplify:

$$\boxed{\begin{aligned} \lambda^{(2)} &= (x^{(0)}, A_1 x^{(1)}) \\ \lambda^{(3)} &= (x^{(1)}, [A_1 - \lambda^{(1)}]x^{(1)}) \end{aligned}}$$

Note: $x^{(2)}$ is gone!

Note that we have $\lambda^{(i)}$ in terms of $\lambda^{(j)}$ and $x^{(j)}$ w/ $j<i$.

This means we solve $[A_0 - \lambda^{(0)}]x^{(0)} = 0$ for $x^{(0)}$ and $\lambda^{(0)}$ then use $x^{(0)}$ to find $\lambda^{(1)}$ then use $\lambda^{(1)}$ and $x^{(0)}$ to find $x^{(1)}$, and so on...

So this is a method for finding corrections to a single eigenvalue and eigenvector. Of course we can repeat the analysis for other eigenvalues and eigenvectors. But it turns out (as you may guess) that if we know all of the eigenvalues and eigenvectors of A_0, then we can use them as an orthonormal basis, and simplify matters.

Now it would be nice to have "formulas" for the $x^{(i)}$ (as opposed to them as unknowns in equations). Turns out, if we solve for all the eigenvectors and eigenvalues of A_0, then we can do so.

If we had all the eigenvectors of a Hermitian operator A_0, say $\{x_n^{(0)}\}$, then we could expand the perturbations $\{x_m^{(i)}\}$ in terms of these.

So we label the equations w/ m:

$$[A_0-\lambda_m^{(0)}]\,x_m^{(0)} = 0$$

$$[A_0-\lambda_m^{(0)}]\,x_m^{(1)} = -[A_1-\lambda_m^{(1)}]\,x_m^{(0)}$$

$$[A_0-\lambda_m^{(0)}]\,x_m^{(2)} = -[A_1-\lambda_m^{(1)}]\,x_m^{(1)} + \lambda_m^{(2)}x_m^{(0)}$$

$$[A_0-\lambda_m^{(0)}]\,x_m^{(3)} = -[A_1-\lambda_m^{(1)}]\,x_m^{(2)} + \lambda_m^{(2)}x_m^{(1)} + \lambda_m^{(3)}x_m^{(0)}$$

Now, assuming $m\neq n$, we smash the second equation by $x_n^{(0)}$.

$$(x_n^{(0)}, [A_0-\lambda_m^{(0)}]\,x_m^{(1)}) = -(x_n^{(0)}, [A_1-\lambda_m^{(1)}]\,x_m^{(0)})$$

$$([\underset{\text{Hermitian}}{A_0}-\overset{\text{real}}{\lambda_m^{(0)}}]^{\dagger}x_n^{(0)}, x_m^{(1)}) = ([A_0-\lambda_m^{(0)}]\,x_n^{(0)}, x_m^{(1)}) = -(x_n^{(0)}, A_1 x_m^{(0)}) + \lambda_m^{(1)}\underbrace{(x_n^{(0)}, x_m^{(0)})}_{=0}$$

since eigenfunctions for nondeg eigenvalues are $\perp$

Then:

$$(\lambda_n^{(0)}-\lambda_m^{(0)})(x_n^{(0)}, x_m^{(1)}) = -(x_n^{(0)}, A_1 x_m^{(0)})$$

hence

$$(x_n^{(0)}, x_m^{(1)}) = \frac{(x_n^{(0)}, A_1 x_m^{(0)})}{\lambda_m^{(0)}-\lambda_n^{(0)}}$$

since no degeneracy, i.e. $\lambda_m^{(0)}\neq\lambda_n^{(0)}$ for $n\neq m$

Using completeness of eigenfunctions $\{x_n^{(0)}\}$ we write $x_m^{(1)} = \sum_n (x_n^{(0)}, x_m^{(1)})\,x_n^{(0)}$

$$\boxed{x_m^{(1)} = \sum_{n\neq m}\frac{(x_n^{(0)}, A_1 x_m^{(0)})}{\lambda_m^{(0)}-\lambda_n^{(0)}}\,x_n^{(0)}}$$

Now let's use the third equation:

$$(x_n^{(0)}, [A_0-\lambda_m^{(0)}]\,x_m^{(2)}) = (x_n^{(0)}, -[A_1-\lambda_m^{(1)}]\,x_m^{(1)} + \lambda_m^{(2)}x_m^{(0)})$$

$$([A_0-\lambda_m^{(0)}]\,x_n^{(0)}, x_m^{(2)}) = -(x_n^{(0)}, A_1 x_m^{(1)}) + \lambda_m^{(1)}(x_n^{(0)}, x_m^{(1)}) + \lambda_m^{(2)}\underbrace{(x_n^{(0)}, x_m^{(0)})}_{=0}$$

$$(x_n^{(0)}, x_m^{(2)}) = \frac{(x_n^{(0)}, A_1 x_m^{(1)}) - \lambda_m^{(1)}(x_n^{(0)}, x_m^{(1)})}{\lambda_m^{(0)}-\lambda_n^{(0)}}$$

$$\boxed{x_m^{(2)} = \sum_{n\neq m}\frac{(x_n^{(0)}, A_1 x_m^{(1)}) - \lambda_m^{(1)}(x_n^{(0)}, x_m^{(1)})}{\lambda_m^{(0)}-\lambda_n^{(0)}}\,x_n^{(0)}}$$

Now why $x_m^{(1)} = \sum_{n \neq m} \frac{(x_n^{(0)}, A_1 x_m^{(0)})}{\lambda_m^{(0)} - \lambda_n^{(0)}} x_n^{(0)}$, which does not include $n=m$.

After all this was derived by expressing $x_m^{(1)} = \sum_n (x_n^{(0)}, x_m^{(1)}) x_n^{(0)}$ over all n.

But of course if $(x_m^{(0)}, x_m^{(1)}) = 0$ then they agree.

Turns out this is an option that can always be taken.

Recall $x_m^{(1)}$ is a solution to: $[A_0 - \lambda_m^{(0)}] x_m^{(1)} = -[A_1 - \lambda_m^{(1)}] x_m^{(0)}$

But so is $y_m^{(1)} = x_m^{(1)} + a x_m^{(0)}$ since $[A_0 - \lambda_m^{(0)}] x_m^{(0)} = 0$.

But then we can just arrange: $(x_m^{(0)}, x_m^{(1)}) = (x_m^{(0)}, y_m^{(1)} - a x_m^{(0)}) = (x_m^{(0)}, y_m^{(1)}) - a$

If we just take $a = (x_m^{(0)}, y_m^{(1)})$ then $(x_m^{(0)}, x_m^{(1)}) = 0$.

since $(x_n^{(0)}, \lambda_n^{(1)} x_n^{(1)}) = 0$

If we take this and revisit: $\lambda_n^{(2)} = (x_n^{(0)}, [A_1 - \lambda_n^{(1)}] x_n^{(1)}) = (x_n^{(0)}, A_1 x_n^{(1)})$

and: $\lambda_n^{(3)} = (x_n^{(0)}, [A_1 - \lambda_n^{(1)}] x_n^{(2)}) - \lambda_n^{(2)} (x_n^{(0)}, x_n^{(1)})$

$= ([A_1 - \lambda_n^{(1)}] x_n^{(0)}, x_n^{(2)})$

but $[A_0 - \lambda^{(0)}] x^{(1)} = -[A_1 - \lambda^{(1)}] x^{(0)}$ $\quad = -([A_0 - \lambda_n^{(0)}] x_n^{(1)}, x_n^{(2)})$

$= -(x_n^{(1)}, [A_0 - \lambda_n^{(0)}] x_n^{(2)})$

but $[A_0 - \lambda^{(0)}] x^{(2)} = -[A_1 - \lambda^{(1)}] x^{(1)} + \lambda^{(2)} x^{(0)}$ $\quad = (x_n^{(1)}, [A_1 - \lambda_n^{(1)}] x_n^{(1)} - \lambda_n^{(2)} x_n^{(0)})$

$= (x_n^{(1)}, [A_1 - \lambda_n^{(1)}] x_n^{(1)})$ since $(x_n^{(1)}, x_n^{(0)}) = 0$

But these are the simplified expressions for $\lambda_n^{(2)}$ and $\lambda_n^{(3)}$ I gave you.

An example: $A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}}_{A_0} + \epsilon \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{A_1}$

Start with the eigens for A_0: $\det(A_0 - \lambda^{(0)} I) = 0 = (\lambda^{(0)} - 1)\left[(\lambda^{(0)} - 1)^2 - 4\right] = (\lambda^{(0)} - 1)(\lambda^{(0)} - 3)(\lambda^{(0)} + 1)$

so $\lambda_1^{(0)} = 1$, $\lambda_2^{(0)} = 3$, $\lambda_3^{(0)} = -1$

$$A_0 x_i^{(0)} = \begin{matrix} a \\ b+2c \\ 2b+c \end{matrix} = \begin{cases} = \begin{matrix} a \\ b \\ c \end{matrix} & \Rightarrow \quad x_1^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ = \begin{matrix} 3a \\ 3b \\ 3c \end{matrix} & \Rightarrow \quad x_2^{(0)} = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \\ = \begin{matrix} -a \\ -b \\ -c \end{matrix} & \Rightarrow \quad x_3^{(0)} = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \end{cases}$$

Using $\lambda_m^{(1)} = (x_m^{(0)}, A_1 x_m^{(0)})$ we find the first order correction to the eigenvalues:

$$\lambda_1^{(1)} = (x_1^{(0)}, A_1 x_1^{(0)}) = (1\ 0\ 0)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (1\ 0\ 0)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\lambda_2^{(1)} = (x_2^{(0)}, A_1 x_2^{(0)}) = (0\ \tfrac{1}{\sqrt{2}}\ \tfrac{1}{\sqrt{2}})\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = (0\ \tfrac{1}{\sqrt{2}}\ \tfrac{1}{\sqrt{2}})\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\lambda_3^{(1)} = (x_3^{(0)}, A_1 x_3^{(0)}) = (0\ \tfrac{1}{\sqrt{2}}\ -\tfrac{1}{\sqrt{2}})\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = (0\ \tfrac{1}{\sqrt{2}}\ -\tfrac{1}{\sqrt{2}})\begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} = 0$$

Using $x_m^{(1)} = \sum_{n \neq m} \frac{(x_n^{(0)}, A_1 x_m^{(0)})}{\lambda_m^{(0)} - \lambda_n^{(0)}} x_n^{(0)}$ we find the first order correction to the eigenvectors:

$$x_1^{(1)} \quad \frac{(x_2^{(0)}, A_1 x_1^{(0)})}{\lambda_1^{(0)} - \lambda_2^{(0)}} x_2^{(0)} + \frac{(x_3^{(0)}, A_1 x_1^{(0)})}{\lambda_1^{(0)} - \lambda_3^{(0)}} x_3^{(0)} = \frac{\frac{1}{\sqrt{2}}}{-2}\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + \frac{-\frac{1}{\sqrt{2}}}{2}\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ -1/2 \\ 0 \end{pmatrix}$$

$$x_2^{(1)} = \frac{(x_1^{(0)}, A_1 x_2^{(0)})}{\lambda_2^{(0)} - \lambda_1^{(0)}} x_1^{(0)} + \frac{(x_3^{(0)}, A_1 x_2^{(0)})}{\lambda_2^{(0)} - \lambda_3^{(0)}} x_3^{(0)} = \frac{\frac{1}{\sqrt{2}}}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{0}{4}\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/2\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

$$x_3^{(1)} = \frac{(x_1^{(0)}, A_1 x_3^{(0)})}{\lambda_3^{(0)} - \lambda_1^{(0)}} x_1^{(0)} + \frac{(x_2^{(0)}, A_1 x_3^{(0)})}{\lambda_3^{(0)} - \lambda_2^{(0)}} x_2^{(0)} = \frac{-\frac{1}{\sqrt{2}}}{-2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{0}{-4}\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/2\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

With these in hand we can now find quadratic and cubic contributions to $\lambda_m^{(0)}$:

$\lambda_1^{(2)} = (x_1^{(0)}, A_1 x_1^{(1)}) = 0 \qquad \lambda_1^{(3)} = (x_1^{(1)}, [A_1 - \lambda_1^{(1)}]\, x_1^{(1)}) = (x_1^{(1)}, A_1 x_1^{(1)}) = 0$

$\lambda_2^{(2)} = (x_2^{(0)}, A_1 x_2^{(1)}) = \frac{1}{4} \qquad \lambda_2^{(3)} = (x_2^{(1)}, [A_1 - \lambda_2^{(1)}]\, x_2^{(1)}) = (x_2^{(1)}, A_1 x_2^{(1)}) = 0$

$\lambda_3^{(2)} = (x_3^{(0)}, A_1 x_3^{(1)}) = -\frac{1}{4} \qquad \lambda_3^{(3)} = (x_3^{(1)}, [A_1 - \lambda_3^{(1)}]\, x_3^{(1)}) \underset{\uparrow}{=} (x_3^{(1)}, A_1 x_3^{(1)}) = 0$

since all 3 $\lambda^{(1)}$'s $= 0$

So in the end, for $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \epsilon \\ 0 & 1 & 2 \\ \epsilon & 2 & 1 \end{pmatrix}$ we have:

$$\lambda_1 = 1 + 0 \cdot \epsilon + 0 \cdot \epsilon^2 + 0 \cdot \epsilon^3 + \cdots \qquad x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ -1/2 \\ 0 \end{pmatrix} + \cdots$$

$$\lambda_2 = 3 + 0 \cdot \epsilon + \frac{1}{4} \epsilon^2 + 0 \cdot \epsilon^3 + \cdots \qquad x_2 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} + \epsilon \begin{pmatrix} 1/2\sqrt{2} \\ 0 \\ 0 \end{pmatrix} + \cdots$$

$$\lambda_3 = -1 + 0 \cdot \epsilon - \frac{1}{4} \epsilon^2 + 0 \cdot \epsilon^3 + \cdots \qquad x_3 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} + \epsilon \begin{pmatrix} 1/2\sqrt{2} \\ 0 \\ 0 \end{pmatrix} + \cdots$$

Now we can actually check the results:

$$\det(A - \lambda I) = 0 = (1-\lambda)\left[(1-\lambda)^2 - 4\right] - \epsilon^2 (1-\lambda) = (1-\lambda)\left[\lambda^2 - 2\lambda - (3+\epsilon^2)\right]$$

But this means that $\lambda = 1$ is an eigenvalue, hence the absence of ϵ-corrections!

For the other two:

$$\lambda_\pm = \frac{2 \pm \sqrt{4 + 4(3+\epsilon^2)}}{2} = 1 \pm \sqrt{4 + \epsilon^2} \Rightarrow \lambda_\pm = \lambda_{\epsilon=0} + \left.\frac{d\lambda}{d\epsilon}\right|_{\epsilon=0} \epsilon + \frac{1}{2} \left.\frac{d^2\lambda}{d\epsilon^2}\right|_{\epsilon=0} \epsilon^2 + \frac{1}{6} \left.\frac{d^3\lambda}{d\epsilon^3}\right|_{\epsilon=0} \epsilon^3 + \cdots$$

$$\lambda_+ = 3 + 0 \cdot \epsilon + \frac{1}{4} \epsilon^2 + 0 \cdot \epsilon^3 + \cdots$$

$$\lambda_- = -1 + 0 \cdot \epsilon - \frac{1}{4} \epsilon^2 + 0 \cdot \epsilon^3 + \cdots$$