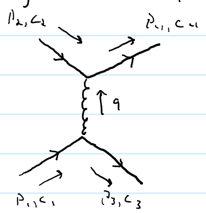
1. Starting with a triplet state, e.g. $\frac{1}{\sqrt{2}}(rb - br)$ we have: $f = \frac{1}{4}(c_3^\dagger \lambda^\alpha c_1)(c_4^\dagger \lambda^\alpha c_2)$ Note: $4 \leftrightarrow 2$ is different than $u\bar{u}$ case.



$$
\begin{aligned}
&= \tfrac{1}{8}\Big[(1\,0\,0)\,\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix}(0\,1\,0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix} \\
&\quad - (1\,0\,0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix}(0\,1\,0)\,\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix} \\
&\quad - (0\,1\,0)\,\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix}(1\,0\,0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix} \\
&\quad + (0\,1\,0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix}(1\,0\,0)\,\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix}\Big] \\
&= \tfrac{1}{8}\left[\lambda^\alpha_{11}\lambda^\alpha_{22} - \lambda^\alpha_{12}\lambda^\alpha_{21} - \lambda^\alpha_{21}\lambda^\alpha_{12} + \lambda^\alpha_{22}\lambda^\alpha_{11}\right] \\
&= \tfrac{1}{8}\left[2\left(\lambda^\alpha_{11}\lambda^\alpha_{22} - \lambda^\alpha_{12}\lambda^\alpha_{21}\right)\right] \\
&= \tfrac{1}{8}\left[2\left(\lambda^3_{11}\lambda^3_{22} + \lambda^8_{11}\lambda^8_{22} - \lambda^1_{12}\lambda^1_{21} - \lambda^2_{12}\lambda^2_{21}\right)\right] \\
&= \tfrac{1}{8}\left[2\left(-1 + \tfrac{1}{3} - 1 - 1\right)\right] \\
&= -\tfrac{2}{3}
\end{aligned}
$$

If the state is $\frac{1}{\sqrt{2}}(rb - br)$ then this describes both 1+2 as well as 3+4, so we have:

1	2	3	4
r	b	r	b
- b	r	r	b
- r	b	b	r
b	r	b	r

and recall $c(r) = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$c(b) = \begin{pmatrix}0\\1\\0\end{pmatrix}$

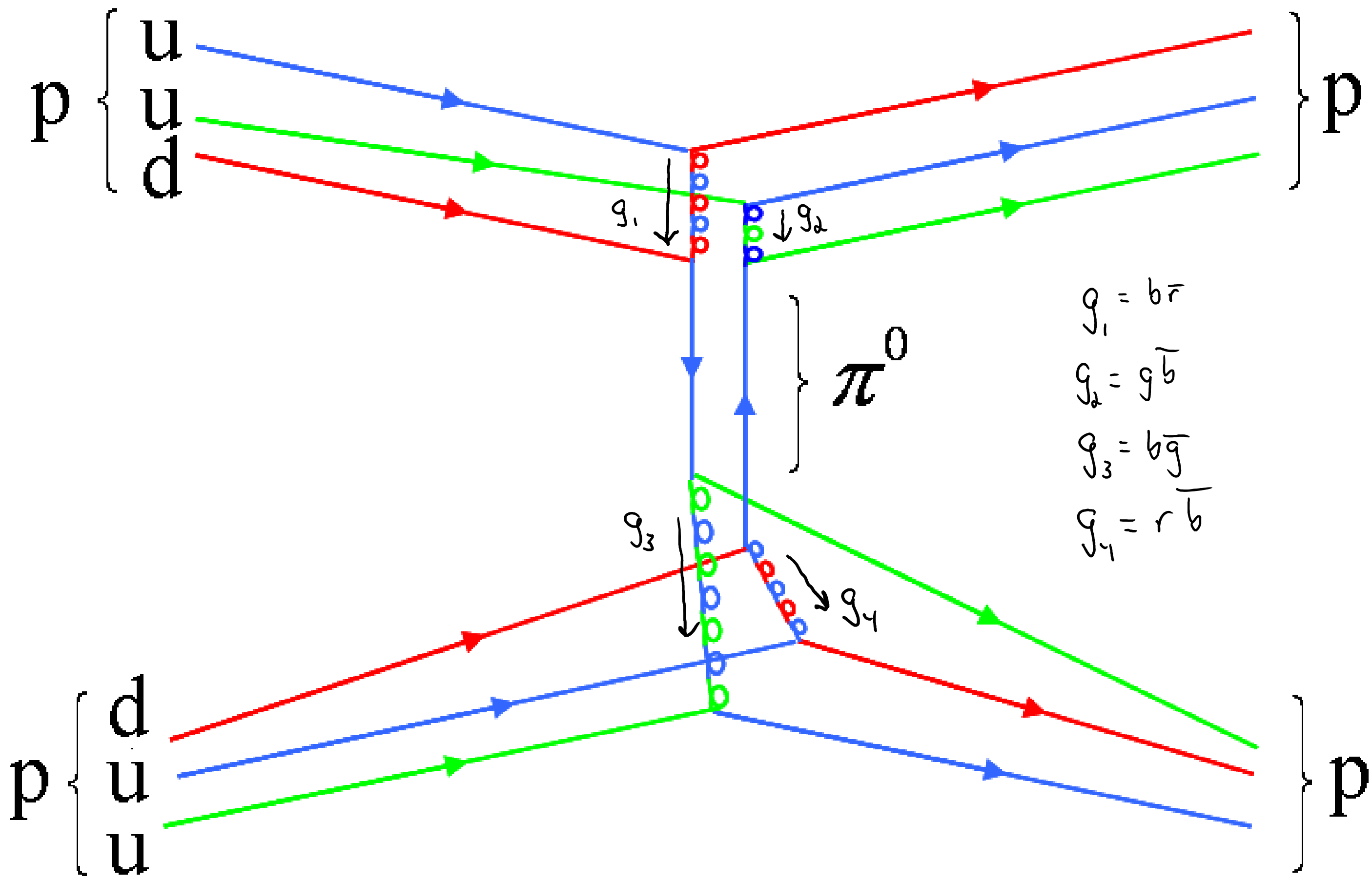
Then using a sextet state, e.g. $\frac{1}{\sqrt{2}}(rb + br)$ we have: $f = \frac{1}{4}(c_3^\dagger \lambda^\alpha c_1)(c_4^\dagger \lambda^\alpha c_2)$

$\frac{1}{3} + 1 = \frac{4}{3}$

1	2	3	4
r	b	r	b
b	r	r	b
r	b	b	r
b	r	b	r

$$
\begin{aligned}
&= \tfrac{1}{8}\Big[(1\,0\,0)\,\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix}(0\,1\,0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix} \\
&\quad + (1\,0\,0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix}(0\,1\,0)\,\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix} \\
&\quad + (0\,1\,0)\,\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix}(1\,0\,0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix} \\
&\quad + (0\,1\,0)\,\lambda^\alpha \begin{pmatrix}0\\1\\0\end{pmatrix}(1\,0\,0)\,\lambda^\alpha \begin{pmatrix}1\\0\\0\end{pmatrix}\Big] \\
&= \tfrac{1}{8}\left[\lambda^\alpha_{11}\lambda^\alpha_{22} + \lambda^\alpha_{12}\lambda^\alpha_{21} + \lambda^\alpha_{21}\lambda^\alpha_{12} + \lambda^\alpha_{22}\lambda^\alpha_{11}\right] \\
&= \tfrac{1}{8}\left[2\left(\lambda^\alpha_{11}\lambda^\alpha_{22} + \lambda^\alpha_{12}\lambda^\alpha_{21}\right)\right] \\
&= \tfrac{1}{8}\left[2\left(\lambda^3_{11}\lambda^3_{22} + \lambda^8_{11}\lambda^8_{22} + \lambda^1_{12}\lambda^1_{21} + \lambda^2_{12}\lambda^2_{21}\right)\right] \\
&= \tfrac{1}{8}\left[2\left(-1 + \tfrac{1}{3} + 1 + 1\right)\right] \\
&= \tfrac{1}{3}
\end{aligned}
$$

2.



$g_1 = b\bar{r}$

$g_2 = g\bar{b}$

$g_3 = b\bar{g}$

$g_4 = r\bar{b}$

3. a) Consider $U = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$ where all elements are complex (18 real parameters) and $U^\dagger U = I$.

$$\begin{pmatrix} a^* & d^* & g^* \\ b^* & e^* & h^* \\ c^* & f^* & j^* \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow$$

$a^*a + d^*d + g^*g = 1 \Rightarrow$ 1 real equation

$a^*b + d^*e + g^*h = 0 \Rightarrow$ 2 real equations

$a^*c + d^*f + g^*j = 0 \Rightarrow$ 2 real equations

$b^*a + e^*d + h^*g = 0$ same as above

$b^*b + e^*e + h^*h = 1 \Rightarrow$ 1 real equation

$b^*c + e^*f + h^*j = 0 \Rightarrow$ 2 real equations

$c^*a + f^*d + j^*g = 0$ same as above

$c^*b + f^*e + j^*h = 0$ same as above

$c^*c + f^*f + j^*j = 1 \Rightarrow$ 1 real equation

18 parameters − 9 real constraints = 9 real free parameters (Recall $SU(N)$ has N^2-1, but here we do not require $\det U = +1$)

b) If we could make U real then $U^\dagger U = 1 \Rightarrow U^T U = 1$ which is the orthogonality condition. In this case we get a set of equations similar to those above, but all of the equations are real. So in the end we have 6 real constraints. A real 3×3 matrix has 9 elements, so this leaves $9-6=3$ free parameters (same as $SO(3)$ for which $\det U = +1$ does not pose a "continuous" constraint.

So if we could make the CKM matrix real by redefining quark phases, we must be able to remove 6 of the free parameters, leaving only the 3 of $O(3)$.

But we only have 3 quark phases and 1 overall phase, hence $9-4=5 \neq 3$ so its impossible to make U real.

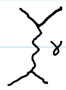
4. a) For $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ w/ complex elements and $U^\dagger U = I$ we have:

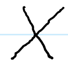
$$\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{array}{ll} a^*a + c^*c = 1 & \Rightarrow 1 \\ \left[\begin{array}{l} a^*b + c^*d = 0 \\ b^*a + d^*c = 0 \end{array} \right. & \begin{array}{l} \Rightarrow 2 \\ \text{same} \end{array} \\ b^*b + d^*d = 1 & = 1 \end{array}$$

8 real elements − 4 constraints = 4 free parameters

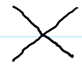
b) To make U real, i.e. $U^T U = I \Rightarrow 3$ constraints $\Rightarrow$ 1 free parameter

But we have 2 quark phases and 1 overall phase so this <u>is</u> possible.

5. The compression of 

into  is not allowed since $m_\gamma = 0$.

Recall that 

can be compressed to  for processes of momentum transfer $q \ll m_W c$.

6. The expressions $\tau \to \tau + 1$ and $\tau \to -\frac{1}{\tau}$ can be combined, each done any number of times to form $\tau \to \tau' = \frac{a\tau + b}{c\tau + d}$ where a, b, c, d are integers.

The fundamental region for this set of transformations is $-\frac{1}{2} \le \mathrm{Re}\,\tau \le \frac{1}{2}$, $|\tau| \ge 1$.

Recall that $\tau = \tau_1 + i\tau_2$ where τ_1, τ_2 are real. This means that $-\frac{1}{2} \le \tau_1 \le \frac{1}{2}$, and since $|\tau| = \tau_1^2 + \tau_2^2 \ge 1$ or $\tau_2^2 \ge 1 - \tau_1^2$, then the smallest value τ_2 can take is when $\tau_1 = \pm \frac{1}{2}$, i.e. $\tau_{2,min} = \sqrt{\frac{3}{4}}$.

But recall that τ_2 is the parameter describing the radius of the strings trajectory. Therefore an $R \to 0$ trajectory is not allowed.