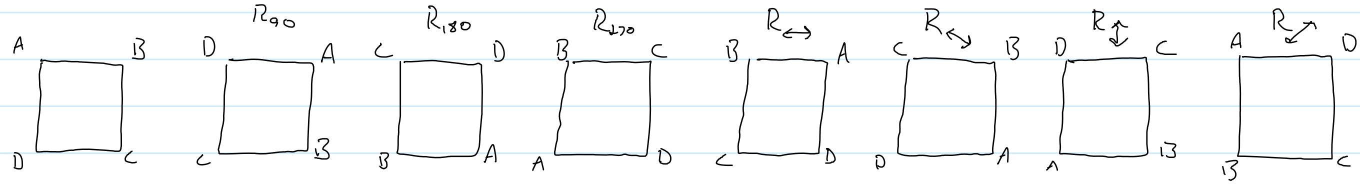
1.

Notice that these are related by R_{90}, R_{180}, R_{270}



$$\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix}$$

There are 8 elements of the group:

$$I = \begin{pmatrix}1&&&&&&&\\&1&&&&&&\\&&1&&&&&\\&&&1&&&&\\&&&&1&&&\\&&&&&1&&\\&&&&&&1&\\&&&&&&&1\end{pmatrix} \quad R_{90} = \begin{pmatrix}0001&\\1000&\\0100&\\0010&\\&0001\\&1000\\&0100\\&0010\end{pmatrix} \quad R_{180} = \begin{pmatrix}0010&\\0001&\\1000&\\0100&\\&0010\\&0001\\&1000\\&0100\end{pmatrix} \quad R_{270} = \begin{pmatrix}0100&\\0010&\\0001&\\1000&\\&0100\\&0010\\&0001\\&1000\end{pmatrix}$$

$R_{180} = R_{90}R_{90}$, $\quad R_{270} = R_{90}R_{90}R_{90}$

$$R_{\leftrightarrow} = \begin{pmatrix}0&1000\\&0001\\&0010\\&0100\\1000&0\\0001&\\0010&\\0100&\end{pmatrix} \quad R_{\nwarrow\searrow} = \begin{pmatrix}0&0100\\&1000\\&0001\\&0010\\0100&0\\1000&\\0001&\\0010&\end{pmatrix} \quad R_{\updownarrow} = \begin{pmatrix}0&0010\\&0100\\&1000\\&0001\\0010&0\\0100&\\1000&\\0001&\end{pmatrix} \quad R_{\swarrow\nearrow} = \begin{pmatrix}0&0001\\&0010\\&0100\\&1000\\0001&0\\0010&\\0100&\\1000&\end{pmatrix}$$

$R_{\nwarrow\searrow} = R_{90}R_{\leftrightarrow}$, $\quad R_{\updownarrow} = R_{90}R_{90}R_{\leftrightarrow}$, $\quad R_{\swarrow\nearrow} = R_{90}R_{90}R_{90}R_{\leftrightarrow}$

Note: If you just found R_{90} and $R_{\leftrightarrow}$, you could build all the rest w/ matrix multiplication!

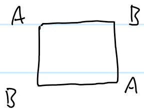
The multiplication table is:

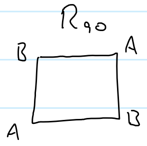
(Second)

(First)	I	R_{90}	R_{180}	R_{270}	$R_{\leftrightarrow}$	$R_{\nwarrow\searrow}$	$R_{\updownarrow}$	$R_{\swarrow\nearrow}$
I	I	R_{90}	R_{180}	R_{270}	$R_{\leftrightarrow}$	$R_{\nwarrow\searrow}$	$R_{\updownarrow}$	$R_{\swarrow\nearrow}$
R_{90}	R_{90}	R_{180}	R_{270}	I	$R_{\swarrow\nearrow}$	$R_{\leftrightarrow}$	$R_{\nwarrow\searrow}$	$R_{\updownarrow}$
R_{180}	R_{180}	R_{270}	I	R_{90}	$R_{\updownarrow}$	$R_{\swarrow\nearrow}$	$R_{\leftrightarrow}$	$R_{\nwarrow\searrow}$
R_{270}	R_{270}	I	R_{90}	R_{180}	$R_{\nwarrow\searrow}$	$R_{\updownarrow}$	$R_{\swarrow\nearrow}$	$R_{\leftrightarrow}$
$R_{\leftrightarrow}$	$R_{\leftrightarrow}$	$R_{\nwarrow\searrow}$	$R_{\updownarrow}$	$R_{\nwarrow\searrow}$	I	R_{90}	R_{180}	R_{270}
$R_{\nwarrow\searrow}$	$R_{\nwarrow\searrow}$	$R_{\updownarrow}$	$R_{\swarrow\nearrow}$	$R_{\leftrightarrow}$	R_{270}	I	R_{90}	R_{180}
$R_{\updownarrow}$	$R_{\updownarrow}$	$R_{\swarrow\nearrow}$	$R_{\leftrightarrow}$	$R_{\nwarrow\searrow}$	R_{180}	R_{270}	I	R_{90}
$R_{\swarrow\nearrow}$	$R_{\swarrow\nearrow}$	$R_{\leftrightarrow}$	$R_{\nwarrow\searrow}$	$R_{\updownarrow}$	R_{90}	R_{180}	R_{270}	I

Not symmetric across the diagonal so this is non-abelian. This should be expected since we are now using 3D rotations which we generally expect to not commute!

If instead we started with:





These are the only two distinct configurations.

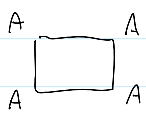
(1) $\quad$ (-1)

Let $I = 1$, $R_{90} = -1$ then

	I	R_{90}
I	I	R_{90}
R_{90}	R_{90}	I

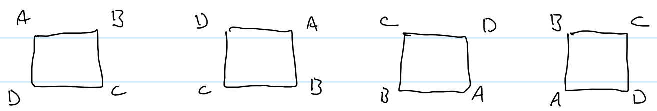
and it is an abelian representation.

Last:



This is the scalar or singlet representation. All transformations just act like I.

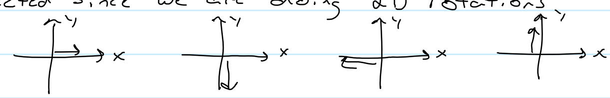
2. Going back to 2D rotations of the square:



$\begin{pmatrix}1\\0\end{pmatrix}$ $\begin{pmatrix}0\\-1\end{pmatrix}$ $\begin{pmatrix}-1\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\end{pmatrix}$

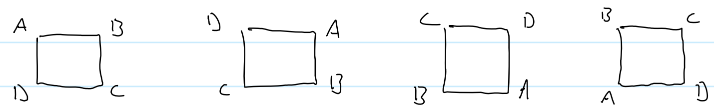
Then: $I = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ $R_{90} = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$ $R_{180} = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$ $R_{270} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$

That we can get a 2D representation might be expected since we are doing 2D rotations, and one way to illustrate 2D rotations is:



(1,0) (0,-1) (-1,0) (0,1)

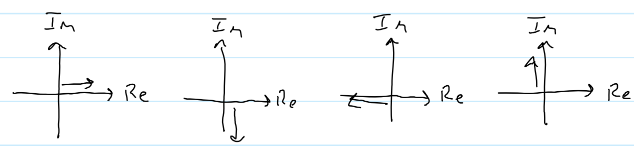
3. For a 1D representation we need complex numbers:



(1) (−i) (−1) (i)

Then: $I = 1$, $R_{90} = -i$, $R_{180} = -1$, $R_{270} = i$

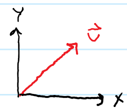
You can think of this as using the 2D complex plane:



1 −i −1 i

4. Consider $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ vs. $B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

If we start w/ a vector of length 1 at $45°$ i.e.



Note: $v_x = \cos 45° = \frac{1}{\sqrt{2}}$, $v_y = \frac{1}{\sqrt{2}}$

Now an active rotation by $\theta = 90°$, is an operation which leaves the coordinate system unchanged and rotates the vector counterclockwise by $90°$, i.e.

Note: $v_x' = -\frac{1}{\sqrt{2}}$, $v_y' = \frac{1}{\sqrt{2}}$

Whereas a passive rotation by $\theta = 90°$ leaves the vector fixed and rotates the coordinate system counterclockwise by $90°$, i.e.

Note: $v_x'' = \frac{1}{\sqrt{2}}$, $v_y'' = -\frac{1}{\sqrt{2}}$

Now we can use the matrices w/ $\theta = 90°$ to determine which role they play.

$A(90°)\vec{v} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ passive

$B(90°)\vec{v} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ active

Again, in this class we largely use passive transformations since vectors actually exist, even though coordinates don't. So it should not matter what coordinates we use, i.e. physics should be invariant under coordinate changes.

5. For $SO(1,2)$ consider $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ with 9 real parameters.

Then $\Lambda^T g \Lambda = g$ gives $\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

$$\begin{pmatrix} -a & d & g \\ -b & e & h \\ -c & f & i \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

Then multiplying and matching left and right we find:

These pairs are the same

$$\begin{array}{ll}
-a^2 + d^2 + g^2 = -1 & 1 \\
-ab + de + gh = 0 & 1 \\
-ac + df + gi = 0 & 1 \\
-ba + ed + hg = 0 & \\
-b^2 + e^2 + h^2 = 1 & 1 \\
-bc + ef + hi = 0 & 1 \\
-ca + fd + ig = 0 & \\
-cb + fe + ih = 0 & \\
-c^2 + f^2 + i^2 = 1 & \underline{\quad 1 \quad} \\
 & 6 \text{ equations}
\end{array}$$

So $\Lambda^T g \Lambda = g$ removes 6 free parameters leaving 3.

For $\det \Lambda = +1$ consider

$$\det(\Lambda^T g \Lambda) = \det g$$
$$\det \Lambda^T \det g \det \Lambda = -1$$
$$\det \Lambda \, (-1) \det \Lambda = -1$$
$$(\det \Lambda)^2 = 1$$
$$\det \Lambda = \pm 1$$

2 discrete solutions so no more parameters lost.