1. $A \to B + C$

Express E_B, E_C, p_B $(= |\vec{p}_B|)$, p_C $(= |\vec{p}_C|)$ all in terms of m_A, m_B, m_C.

In the rest frame of A: $P_A^\mu = \begin{pmatrix} m_A c \\ \vec{0} \end{pmatrix}$ and since $P_A^\mu = P_B^\mu + P_C^\mu$

we consider $P_A^\mu - P_B^\mu = P_C^\mu$

It helps to recall that for a single particle
$P_\mu P^\mu = -m^2c^2$
To see this recall that $P_\mu P^\mu$ is invariant
so we can just evaluate it in the particle's
rest frame where $P^\mu = \begin{pmatrix} mc \\ \vec{0} \end{pmatrix} \Rightarrow P_\mu = (-mc \ \ \vec{0})$

then $(P_{A\mu} - P_{B\mu})(P_A^\mu - P_B^\mu) = P_{C\mu} P_C^\mu$

$$P_{A\mu}P_A^\mu - 2P_{A\mu}P_B^\mu + P_{B\mu}P_B^\mu = P_{C\mu}P_C^\mu$$

$$-m_A^2c^2 - 2m_A E_B - m_B^2c^2 = -m_C^2c^2$$

$$E_B = \frac{1}{2m_A}\left[-m_C^2c^2 + m_A^2c^2 + m_B^2c^2\right]$$

and by inspection

$$E_C = \frac{1}{2m_A}\left[-m_B^2c^2 + m_A^2c^2 + m_C^2c^2\right]$$

Then using that: $\frac{E^2}{c^2} - p^2 = m^2c^2 \Rightarrow p^2 = \frac{E^2}{c^2} - m^2c^2$

we find: $$p_B = \sqrt{\frac{E_B^2}{c^2} - m_B^2c^2} = \sqrt{\frac{1}{4m_A^2c^2}\left[-m_C^2c^2 + m_A^2c^2 + m_B^2c^2\right]^2 - m_B^2c^2}$$

and: $$p_C = \sqrt{\frac{E_C^2}{c^2} - m_C^2c^2} = \sqrt{\frac{1}{4m_A^2c^2}\left[-m_B^2c^2 + m_A^2c^2 + m_C^2c^2\right]^2 - m_C^2c^2}$$

It turns out that if you multiply these out, you will find that $p_B = p_C$. But this is as expected since $\vec{p}_{tot} = \vec{0}$ for the decay of a particle at rest.

2. Recall that
$$F_{\mu\nu} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \end{array}$$

so $F_{23} = B_x$ and $F_{22} = 0$, but we want to get these from $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
where $A_\mu = (\phi, \vec{A})$ and $\vec{E} = \vec{\nabla}\phi + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$

Thus: $F_{23} = \partial_2 A_3 - \partial_3 A_2 = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$, but this is the x-component of $\vec{\nabla} \times \vec{A}$, hence B_x.
$F_{22} = \partial_2 A_2 - \partial_2 A_2 = 0$ obviously

3. First of all $g_{R_{xy}} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} = \frac{1}{2}\sigma_3$ where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \sigma_3^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3$, etc.

Now note that $g_{R_{xy}}^2 = \frac{1}{4}\sigma_3^2 = \frac{1}{4} I$ and $g_{R_{xy}}^3 = \frac{1}{8}\sigma_3^3 = \frac{1}{8}\sigma_3$, in general $g_{R_{xy}}^n = \begin{cases} \frac{1}{2^n}\sigma_3 & \text{for } n\text{-odd} \\ \frac{1}{2^n} I & \text{for } n\text{-even} \end{cases}$

Using these we can now evaluate:

$$R_{xy} = e^{i g_{R_{xy}} \theta} = e^{i\frac{1}{2}\sigma_3\theta} = I + \frac{i}{2}\sigma_3\theta - \frac{1}{2!}\frac{1}{4}\sigma_3^2\theta^2 + \frac{1}{3!}(-i)\frac{1}{8}\sigma_3\theta^3 + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{i}{2}\theta & 0 \\ 0 & -\frac{i}{2}\theta \end{pmatrix} + \begin{pmatrix} -\frac{1}{8}\theta^2 & 0 \\ 0 & -\frac{1}{8}\theta^2 \end{pmatrix} + \begin{pmatrix} -\frac{i}{48}\theta^3 & 0 \\ 0 & \frac{i}{48}\theta^3 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} e^{\frac{i}{2}\theta} & 0 \\ 0 & e^{-\frac{i}{2}\theta} \end{pmatrix}$$

4, To get an idea for how this works let's look at the most interesting case:

$$[g_4, g_5] = i f^{45k} g_k$$

Left hand side: $[g_4, g_5] = \left[\frac{\lambda_4}{2}, \frac{\lambda_5}{2}\right] = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -i/2 \\ 0 & 0 & 0 \\ i/2 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -i/2 \\ 0 & 0 & 0 \\ i/2 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} i/4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i/4 \end{pmatrix} - \begin{pmatrix} -i/4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i/4 \end{pmatrix}$$

$$= \begin{pmatrix} i/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i/2 \end{pmatrix}$$

Right hand side: $i f^{45k} g_k = \cancelto{0}{i f^{451}} g_1 + \cancelto{0}{i f^{452}} g_2 + i f^{453} g_3 + \cancelto{0}{i f^{454}} g_4 + \cancelto{0}{i f^{455}} g_5 + \cancelto{0}{i f^{456}} g_6 + \cancelto{0}{i f^{457}} g_7 + i f^{458} g_8$

$= i f^{453} g_3 + i f^{458} g_8$ but from what was given: $f^{345} = \frac{1}{2} \Rightarrow \underbrace{f^{453} = \frac{1}{2}}_{\text{since 453 is cyclic permutation of 345.}}$

$f^{458} = \frac{\sqrt{3}}{2}$

$$= i \frac{1}{2} \frac{1}{2} \lambda_3 + i \frac{\sqrt{3}}{2} \frac{1}{2} \lambda_8$$

$$= \frac{i}{4} \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} + \frac{i\sqrt{3}}{4} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

$$= \begin{pmatrix} i/4 & & \\ & -i/4 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} i/4 & & \\ & i/4 & \\ & & -i/2 \end{pmatrix}$$

$= \begin{pmatrix} i/2 & & \\ & 0 & \\ & & -i/2 \end{pmatrix}$ which agrees w/ l.h.s. ✓

5. $J_1 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-i \\ 0&0&i&0 \end{pmatrix}$ $\quad J_2 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&i \\ 0&0&0&0 \\ 0&-i&0&0 \end{pmatrix}$ $\quad J_3 = \begin{pmatrix} 0&0&0&0 \\ 0&0&-i&0 \\ 0&i&0&0 \\ 0&0&0&0 \end{pmatrix}$

$K_1 = \begin{pmatrix} 0&-i&0&0 \\ -i&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$ $\quad K_2 = \begin{pmatrix} 0&0&-i&0 \\ 0&0&0&0 \\ -i&0&0&0 \\ 0&0&0&0 \end{pmatrix}$ $\quad K_3 = \begin{pmatrix} 0&0&0&-i \\ 0&0&0&0 \\ 0&0&0&0 \\ -i&0&0&0 \end{pmatrix}$

$[J_i, J_j] = i\epsilon^{ijk} J_k$

Check $i=1, j=2$ $\quad [J_1, J_2] = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-i \\ 0&0&i&0 \end{pmatrix}\begin{pmatrix} 0&0&0&0 \\ 0&0&0&i \\ 0&0&0&0 \\ 0&-i&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&0 \\ 0&0&0&i \\ 0&0&0&0 \\ 0&-i&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-i \\ 0&0&i&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&-1&0&0 \\ 0&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&0 \\ 0&0&-1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&-1&0&0 \\ 0&0&0&0 \end{pmatrix}$

$i\epsilon^{123} J_3 = i(+1)\begin{pmatrix} 0&0&0&0 \\ 0&0&-i&0 \\ 0&i&0&0 \\ 0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&-1&0&0 \\ 0&0&0&0 \end{pmatrix}$

Same ✓

$[K_i, K_j] = -i\epsilon^{ijk} J_k$

Check $i=1, j=2$ $\quad [K_1, K_2] = \begin{pmatrix} 0&-i&0&0 \\ -i&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&-i&0 \\ 0&0&0&0 \\ -i&0&0&0 \\ 0&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&-i&0 \\ 0&0&0&0 \\ -i&0&0&0 \\ 0&0&0&0 \end{pmatrix}\begin{pmatrix} 0&-i&0&0 \\ -i&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0 \\ 0&0&-1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&-1&0&0 \\ 0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0 \\ 0&0&-1&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{pmatrix}$

$-i\epsilon^{123} J_3 = -i(+1)\begin{pmatrix} 0&0&0&0 \\ 0&0&-i&0 \\ 0&i&0&0 \\ 0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0 \\ 0&0&-1&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{pmatrix}$

Same ✓

$[J_i, K_j] = i\epsilon^{ijk} K_k$

Check $i=1, j=2$ $\quad [J_1, K_2] = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-i \\ 0&0&i&0 \end{pmatrix}\begin{pmatrix} 0&0&-i&0 \\ 0&0&0&0 \\ -i&0&0&0 \\ 0&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&-i&0 \\ 0&0&0&0 \\ -i&0&0&0 \\ 0&0&0&0 \end{pmatrix}\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-i \\ 0&0&i&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&-1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix}$

$i\epsilon^{123} K_3 = i(+1)\begin{pmatrix} 0&0&0&-i \\ 0&0&0&0 \\ 0&0&0&0 \\ -i&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix}$

Same ✓

6. Recall that $J_{\pm i} = \frac{1}{2}(J_i \pm i K_i)$

We will demonstrate:

a) $[J_{+1}, J_{+2}] = \left[\frac{1}{2}(J_1 + iK_1), \frac{1}{2}(J_2 + iK_2)\right] = \frac{1}{4}\left\{[J_1, J_2] - [K_1, K_2] + i[J_1, K_2] + i[K_1, J_2]\right\}$

$= \frac{1}{4}\{iJ_3 + iJ_3 - K_3 - K_3\}$

$= i\,\frac{1}{2}(J_3 + iK_3)$

$= iJ_{+3}$

Note: $[J_i, K_j] = i\epsilon^{ijk}K_k \Rightarrow [J_1, K_2] = iK_3$

$[K_i, J_j] = -[J_j, K_i] = -i\epsilon^{jik}K_k$

$[K_1, J_2] = -i\epsilon^{213}K_3 = iK_3$

b) $[J_{+1}, J_{-2}] = \left[\frac{1}{2}(J_1 + iK_1), \frac{1}{2}(J_2 - iK_2)\right] = \frac{1}{4}\left\{[J_1, J_2] + [K_1, K_2] - i[J_1, K_2] + i[K_1, J_2]\right\}$

$= \frac{1}{4}\{iJ_3 - iJ_3 + K_3 - K_3\}$

$= 0$