1. a) $\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}$

Use: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \Rightarrow \mathrm{Tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = 2\eta^{\mu\nu}\,\mathrm{Tr}\,I = 8\eta^{\mu\nu}$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) + \mathrm{Tr}(\gamma^\nu\gamma^\mu) = 8\eta^{\mu\nu}$$
$$2\,\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 8\eta^{\mu\nu}$$
$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu} \quad \checkmark$$

b) $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho) = 4(\eta^{\mu\nu}\eta^{\lambda\rho} - \eta^{\mu\lambda}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\lambda})$

Use: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \Rightarrow \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu + 2\eta^{\mu\nu}$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho) = \mathrm{Tr}(-\gamma^\nu\gamma^\mu\gamma^\lambda\gamma^\rho) + \mathrm{Tr}(2\eta^{\mu\nu}\gamma^\lambda\gamma^\rho)$$
$$= -\mathrm{Tr}(\gamma^\nu\gamma^\mu\gamma^\lambda\gamma^\rho) + 2\eta^{\mu\nu}\,\mathrm{Tr}(\gamma^\lambda\gamma^\rho)$$
$$= -\mathrm{Tr}(\gamma^\nu\gamma^\mu\gamma^\lambda\gamma^\rho) + 8\eta^{\mu\nu}\eta^{\lambda\rho}$$
$$= -\mathrm{Tr}(\gamma^\rho\gamma^\nu\gamma^\mu\gamma^\lambda) + 8\eta^{\mu\nu}\eta^{\lambda\rho}$$
$$= \mathrm{Tr}(\gamma^\rho\gamma^\nu\gamma^\lambda\gamma^\mu) - 2\eta^{\mu\lambda}\,\mathrm{Tr}(\gamma^\rho\gamma^\nu) + 8\eta^{\mu\nu}\eta^{\lambda\rho}$$
$$= \mathrm{Tr}(\gamma^\nu\gamma^\lambda\gamma^\mu\gamma^\rho) - 8\eta^{\mu\lambda}\eta^{\rho\nu} + 8\eta^{\mu\nu}\eta^{\lambda\rho}$$
$$= -\mathrm{Tr}(\gamma^\nu\gamma^\lambda\gamma^\rho\gamma^\mu) + 2\eta^{\rho\mu}\,\mathrm{Tr}(\gamma^\nu\gamma^\lambda) - 8\eta^{\mu\lambda}\eta^{\rho\nu} + 8\eta^{\mu\nu}\eta^{\lambda\rho}$$
$$2\,\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho) = 8\eta^{\rho\mu}\eta^{\nu\lambda} - 8\eta^{\mu\lambda}\eta^{\rho\nu} + 8\eta^{\mu\nu}\eta^{\lambda\rho}$$
$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho) = 4(\eta^{\mu\nu}\eta^{\lambda\rho} - \eta^{\mu\lambda}\eta^{\rho\nu} + \eta^{\mu\rho}\eta^{\nu\lambda}) \quad \checkmark$$

c) $\gamma^5\gamma^5 = 1$

$(-i\gamma^0\gamma^1\gamma^2\gamma^3) \equiv \gamma^5$ and use $(\gamma^0)^2 = -1$, $(\gamma^i)^2 = +1$ and $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad \mu \neq \nu$

$$\gamma^5\gamma^5 = (-i\gamma^0\gamma^1\gamma^2\gamma^3)(-i\gamma^0\gamma^1\gamma^2\gamma^3) = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^{0\,2}\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 = \gamma^{0\,2}\gamma^{1\,2}\gamma^2\gamma^3\gamma^2\gamma^3$$
$$= -\gamma^{0\,2}\gamma^{1\,2}\gamma^{2\,2}\gamma^{3\,2}$$
$$= 1 \quad \checkmark$$

d) $\{\gamma^\mu, \gamma^5\} = 0$

$$\{\gamma^\mu, \gamma^5\} = -i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 - i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu$$

For $\mu = 0 \quad \{\gamma^0, \gamma^5\} = -i\gamma^{0\,2}\gamma^1\gamma^2\gamma^3 + i\gamma^{0\,2}\gamma^1\gamma^2\gamma^3 = 0$

$\mu = 2 \quad \{\gamma^2, \gamma^5\} = -i\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 - i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 = -i\gamma^0\gamma^1\gamma^{2\,2}\gamma^3 + i\gamma^0\gamma^1\gamma^{2\,2}\gamma^3 = 0$

Similarly for $\mu = 1, 3$.

e) $\mathrm{Tr}(\gamma^\mu\gamma^\nu \cdots) = 0$ for odd # of γ's

Recall that $\gamma^5\gamma^5 = 1$ then $\mathrm{Tr}(\gamma^\mu) = \mathrm{Tr}(\gamma^\mu\gamma^5\gamma^5) = \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^5) = -\mathrm{Tr}(\gamma^\mu\gamma^5\gamma^5)$
and $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$

But if $\mathrm{Tr}(\gamma^\mu\gamma^5\gamma^5) = -\mathrm{Tr}(\gamma^\mu\gamma^5\gamma^5)$ then $\mathrm{Tr}(\gamma^\mu\gamma^5\gamma^5) = \mathrm{Tr}(\gamma^\mu) = 0$

This argument can be iterated for higher odd numbers of γ's.

f) $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = -\mathrm{Tr}(\gamma^5\gamma^\nu\gamma^\mu)$

$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma^\nu\gamma^5\gamma^\mu) = \mathrm{Tr}(\underbrace{\gamma^5\gamma^5}_{=1}\gamma^\nu\gamma^5\gamma^\mu) = -\mathrm{Tr}(\gamma^5\gamma^\nu\gamma^5\gamma^5\gamma^\mu) = -\mathrm{Tr}(\gamma^5\gamma^\nu\gamma^\mu)$ ✓

2. We want to show that $\left(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\right)^{\dagger}\gamma^0 = \gamma^0 e^{-\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}$ where $\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu]$

Expanding: $\left(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\right)^{\dagger}\gamma^0 = \left(1 + \frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu} + \frac{1}{2!}\left(\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}\right)^2 + \cdots\right)^{\dagger}\gamma^0$

To move γ^0 to the left, we can freely move it across everything except $\sigma^{\mu\nu}$.
Since $\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu] = \frac{i}{4}\gamma^\mu\gamma^\nu - \frac{i}{4}\gamma^\nu\gamma^\mu$ and $\gamma^i\gamma^0 = -\gamma^0\gamma^i$
We find: $\sigma^{ij}\gamma^0 = \gamma^0\sigma^{ij}$ since γ^0 moves across 2 γ's
$\sigma^{0i}\gamma^0 = -\gamma^0\sigma^{0i}$ since γ^0 moves across 1 γ

So moving γ^0 to the left depends on whether we have σ^{ij} or σ^{0i} (note $\sigma^{00} = 0$).
But when we hermitian conjugate, the result also depends on the $\sigma^{\mu\nu}$, i.e. $\sigma^{ij\dagger} = \sigma^{ij}$, $\sigma^{0i\dagger} = -\sigma^{0i}$.

So when we do both, the negative sign from moving γ^0 left across σ^{0i} cancels the negative from conjugating.

Then: $\left(1 + \frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu} + \frac{1}{2!}\left(\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}\right)^2 + \cdots\right)^{\dagger}\gamma^0 = \gamma^0\left(1 - \frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu} + \frac{1}{2!}\left(\frac{-i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}\right)^2 + \cdots\right)$

remember $\dagger$ includes $*$

Or: $\gamma^0\left(e^{-\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\right)$ ✓

3. For $G[f(x)] = \int_0^1 \overbrace{\left(\underbrace{(f-1)^2}_{\geq 0} + \underbrace{f'^2}_{\geq 0} \right)}^{L(x)} dx$ so we expect a minimum when $\left.\begin{array}{l} f-1=0 \\ f'=0 \end{array}\right\} \begin{array}{l} f(x)=1 \\ f'(x)=0 \end{array}$ $f(0)=f(1)=1$

The e.o.m. is: $\frac{\partial L}{\partial f} - \frac{d}{dx}\left(\frac{\partial L}{\partial (f')}\right) = 2(f-1) - \frac{d}{dx} 2f' = 2f - 2 - 2f'' = 0 = f - 1 - f''$

For $f(x)=1$ this is clearly satisfied.

4. $\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} A_\mu J^\mu = -\frac{1}{16\pi}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{c} A_\mu J^\mu$

$$= \underbrace{-\frac{1}{16\pi} \eta^{\mu\lambda}\eta^{\nu\rho}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\lambda A_\rho - \partial_\rho A_\lambda)}_{\mathcal{L}_1} \underbrace{- \frac{1}{c} A_\mu J^\mu}_{\mathcal{L}_2}$$

Varying w.r.t. A_μ : $\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\right)$

$\frac{\partial \mathcal{L}}{\partial A_\mu} \to -\frac{1}{c} J^\mu$

To compute $\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}$ start w/ $\mathcal{L}_1$ above and use new letters for all of the dummy indices.

$$\frac{\partial}{\partial(\partial_\nu A_\mu)}\left[-\frac{1}{16\pi}\eta^{\alpha\lambda}\eta^{\beta\rho}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\lambda A_\rho - \partial_\rho A_\lambda)\right]$$

$$= -\frac{1}{16\pi}\eta^{\alpha\lambda}\eta^{\beta\rho}\left[(\delta_\alpha^\nu \delta_\beta^\mu - \delta_\beta^\nu \delta_\alpha^\mu)(\partial_\lambda A_\rho - \partial_\rho A_\lambda) + (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\delta_\lambda^\nu \delta_\rho^\mu - \delta_\rho^\nu \delta_\lambda^\mu)\right]$$

$$= -\frac{1}{16\pi}\left[(\delta_\alpha^\nu \delta_\beta^\mu - \delta_\beta^\nu \delta_\alpha^\mu) F^{\alpha\beta} + F^{\lambda\rho}(\delta_\lambda^\nu \delta_\rho^\mu - \delta_\rho^\nu \delta_\lambda^\mu)\right]$$

$$= -\frac{1}{16\pi}\left[F^{\nu\mu} - F^{\mu\nu} + F^{\nu\mu} - F^{\mu\nu}\right] \quad \text{but } F^{\nu\mu} = -F^{\mu\nu}$$

$$= -\frac{1}{4\pi} F^{\nu\mu}$$

$$= \frac{1}{4\pi} F^{\mu\nu}$$

So in total $\frac{\partial \mathcal{L}_1}{\partial(\partial_\nu A_\mu)} = \frac{1}{4\pi} F^{\mu\nu}$

Altogether: $\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\right) = 0 \Rightarrow -\frac{1}{c} J^\mu - \frac{1}{4\pi}\partial_\nu F^{\mu\nu} = 0 = -\frac{1}{c} J^\mu + \frac{1}{4\pi}\partial_\nu F^{\nu\mu}$

$$\boxed{\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} J^\mu}$$

The best way to understand the lengthy procedure above is as follows.
Suppose we start w/ $\mathcal{L}_1 = -\frac{1}{16\pi}\eta^{\mu\lambda}\eta^{\nu\rho}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\lambda A_\rho - \partial_\rho A_\lambda)$

If we were taking the derivative w.r.t. $\frac{\partial \mathcal{L}_1}{\partial(\partial_0 A_1)}$ we would have to find every place that this term appears. But since every index in sight is summed over, we should expect $\partial_0 A_1$ to appear once in each term when we do the sum over indices.

A simpler example is: $\mathcal{L} = A_\nu A^\nu = \eta^{\mu\nu} A_\nu A_\mu \Rightarrow \frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{\partial(\eta^{\mu\nu} A_\nu A_\mu)}{\partial A_\nu} = \eta^{\mu\nu} A_\mu + \eta^{\nu\mu} A_\mu = 2\eta^{\mu\nu} A_\mu$

Then for example: $\frac{\partial \mathcal{L}}{\partial A_0} = \frac{\partial}{\partial A_0}(-A_0 A_0 + A_1 A_1 + A_2 A_2 + A_3 A_3)$

$= -2A_0$

5. The Dirac equation: $\gamma^\mu \partial_\mu \psi + \frac{mc}{\hbar}\psi = 0$

The Klein-Gordon equation: $\partial_\mu \partial^\mu \psi - \left(\frac{mc}{\hbar}\right)^2 \psi = 0$

Let's go... If $\left(\gamma^\mu \partial_\mu + \frac{mc}{\hbar}\right)\psi = 0$

then $\left(\gamma^\nu \partial_\nu - \frac{mc}{\hbar}\right)\left(\gamma^\mu \partial_\mu + \frac{mc}{\hbar}\right)\psi = 0$

so $\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu \psi - \left(\frac{mc}{\hbar}\right)^2 \psi = 0$

but $-\gamma^\mu \gamma^\nu \partial_\nu \partial_\mu \psi + 2\eta^{\mu\nu} \partial_\nu \partial_\mu \psi - \left(\frac{mc}{\hbar}\right)^2 \psi = 0$

relabelling w/ $\mu \leftrightarrow \nu$

$\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu \psi - \left(\frac{mc}{\hbar}\right)^2 \psi = 0$ adding this line to

these can be freely switched

$2\eta^{\mu\nu} \partial_\mu \partial_\nu \psi - 2\left(\frac{mc}{\hbar}\right)^2 \psi = 0$

or

$\partial_\mu \partial^\mu \psi - \left(\frac{mc}{\hbar}\right)^2 \psi = 0$ Boom!!