1. a) $\text{Tr} \left( Y^wY'^w \right) = \pi^w$

Use: $\left[ Y^w, Y'^w \right] = 2\pi^w \Rightarrow \text{Tr} \left( Y^wY'^w + Y'^wY^w \right) = 2\pi^w \quad \text{Tr} I = 8\pi^w$

$\text{Tr} \left( Y^wY'^w \right) + \text{Tr} \left( Y'^wY^w \right) = 8\pi^w$

$\frac{1}{2} \text{Tr} \left( Y^wY'^w \right) = 4\pi^w \quad \checkmark$

b) $\text{Tr} \left( Y^vY^wY^vY^w \right) = 4 \left( \pi^v, \pi^w, \pi^v - \pi^v \pi^w + \pi^w \pi^v \right)$

Use: $\left[ Y^v, Y^v \right] = 2\pi^v \Rightarrow \pi^v = -\pi^v + 2\pi^v$

$\text{Tr} \left( Y^vY^wY^vY^w \right) = \text{Tr} \left( - Y^vY^wY^vY^w \right) + \text{Tr} \left( 4\pi^v \pi^w \right)$

$= \text{Tr} \left( Y^vY^wY^vY^w \right) + 8\pi^v \pi^w$ \quad $\checkmark$

$= \frac{1}{2} \text{Tr} \left( Y^vY^wY^vY^w \right) + 8\pi^v \pi^w$ \quad $\text{Tr} \left( Y^vY^wY^vY^w \right) = 8 \left( \pi^v, \pi^w, \pi^v - \pi^v \pi^w + \pi^w \pi^v \right)$

$\frac{1}{2} \text{Tr} \left( Y^vY^wY^vY^w \right) = 4 \left( \pi^v, \pi^w, \pi^v - \pi^v \pi^w + \pi^w \pi^v \right) \quad \checkmark$

c) $Y^vY^v = 1$

$\left( -1 ; \ Y^vY^vY^v \right) \equiv Y^v \quad \text{and} \quad \left( Y^v \right)^2 = -1 \left( Y^v \right)^3 = +1 \quad \text{and} \quad Y^vY^v = Y^vY^v \text{ } \forall \text{ } v$

$Y^vY^v = \left( -1 ; Y^vY^vY^v \right) \equiv Y^v$ \quad $\checkmark$

$d) \left[ Y^v, Y^v \right] = 0$

$\left[ Y^v, Y^v \right] = -Y^vY^vY^v - Y^vY^vY^v \quad \forall \text{ } v$

For $v = 0$ \quad $\left[ Y^v, Y^v \right] = -Y^0Y^0Y^0 - Y^0Y^0Y^0 = 0$

For $v = 2$ \quad $\left[ Y^v, Y^v \right] = -Y^2Y^2Y^2 - Y^2Y^2Y^2 = 0$

Similarly for $v = 1,3$.

e) $\text{Tr} \left( Y^wY^w \right) = 0$ for odd # of $Y^w$

Recall that $Y^0Y^0 = 1$ \quad Then $\text{Tr} \left( Y^0 \right) = \text{Tr} \left( Y^0Y^0 \right) = \text{Tr} \left( Y^0 \right)^2 = -\text{Tr} \left( Y^0 \right)^4$

But if $\text{Tr} \left( Y^wY^w \right) = -\text{Tr} \left( Y^wY^w \right)^2$ then $\text{Tr} \left( Y^wY^w \right)^2 = \text{Tr} \left( Y^w \right)^2 = 0$

This argument can be iterated for higher odd numbers of $Y^w$.

$f) \text{Tr} \left( Y^vY^wY^vY^w \right) = -\text{Tr} \left( Y^wY^vY^wY^v \right)$

$\text{Tr} \left( Y^vY^wY^vY^w \right) = \text{Tr} \left( Y^wY^vY^wY^v \right) = \text{Tr} \left( Y^wY^vY^vY^w \right) = -\text{Tr} \left( Y^wY^vY^vY^w \right) = -\text{Tr} \left( Y^vY^wY^vY^w \right) \quad \checkmark$
2. We want to show that \((e^{\frac{i}{2} \sigma \cdot y \cdot \omega})^T \gamma^0 = \gamma^0 e^{-\frac{i}{2} \sigma \cdot y \cdot \omega}\), where \(\sigma^\nu = \frac{i}{2} [\gamma^\nu, \gamma^0]\)

Expanding: \((e^{\frac{i}{2} \sigma \cdot y \cdot \omega})^T \gamma^0 = \left(1 + \frac{i}{2} \sigma \cdot y \cdot \omega + ...\right) \gamma^0\)

To move \(\gamma^0\) to the left, we can freely move it across everything except \(\sigma^\nu\).

Since \(\sigma^\nu = \frac{i}{2} [\gamma^\nu, \gamma^0] = \frac{i}{2} \gamma^\nu \gamma^0 - \frac{i}{2} \gamma^0 \gamma^\nu\) and \(\gamma^0 \gamma^\nu = -\gamma^\nu \gamma^0\),

We find: \(\sigma^\nu \gamma^0 = \gamma^0 \sigma^\nu\), since \(\gamma^0\) moves across \(\gamma^\nu\).

So \(\sigma^\nu \gamma^0 = -\gamma^0 \sigma^\nu\), since \(\gamma^0\) moves across \(\gamma^\nu\).

So moving \(\gamma^0\) to the left depends on whether we have \(\sigma^\nu \gamma^0\) or \(\gamma^0 \sigma^\nu\) (note \(\sigma^\nu \gamma^0 = 0\)).

But when we horizontal conjugate, the result also depends on the \(\sigma^\nu\), i.e. \(\sigma^\nu = \sigma^\nu\), \(\sigma^\nu \gamma^0 = -\gamma^0 \sigma^\nu\).

So when we do both, the negative sign from moving \(\gamma^0\) left across \(\sigma^\nu\) cancels the negative from conjugating.

\(\prod_{\nu} \left(1 + \frac{i}{2} \sigma \cdot y \cdot \omega + ...\right) \gamma^0 = \gamma^0 \left(1 - \frac{i}{2} \sigma \cdot y \cdot \omega + ...\right)\)

Or: \(\gamma^0 (e^{-\frac{i}{2} \sigma \cdot y \cdot \omega})^T\)

[Remember + includes *]
3. For $G[f(x)] = \int_0^1 \left( (f(x)-f)^2 + f^2 \right) \, dx$ 

we expect a minimum when

\[
\begin{align*}
-f' & = 0 \\
f & = 0 \\
f' & = 0
\end{align*}
\]

The c.e.m. is:

\[
\frac{dt}{dt} - \frac{d}{dx} \left( \frac{dt}{dG} \right) = 2(f(x)-f) - \frac{d}{dt} f' = 2f - 2f' = 0 \implies f = \frac{f}{f} - f
\]

For $f(x) = 1$ this is clearly satisfied.
\[ \mathcal{L} = -\frac{1}{16\pi} \varepsilon_{\mu
u} F^{\mu\nu} - \frac{2}{3} A_m \tilde{A}^m = -\frac{1}{16\pi} \varepsilon_{\mu
u} \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right) - \frac{2}{3} A_m \tilde{A}^m \]

Varying w.r.t. \( A_m \): \[ \frac{\partial \mathcal{L}}{\partial A_m} - \partial_\mu \left( \frac{\partial F^{\mu\nu}}{\partial A_m} \right) = -\frac{1}{3} \tilde{A}^m \]

To compute \( \frac{\partial F^{\mu\nu}}{\partial A_m} \) start w/ \( \mathcal{L} \) above and use new letters for all of the dummy indices.

\[ \frac{\partial}{\partial A_m} \left[ -\frac{1}{16\pi} \varepsilon_{\mu\nu\lambda\kappa} \partial_\lambda F^{\kappa\nu} \right] = -\frac{1}{16\pi} \varepsilon_{\mu\nu\lambda\kappa} \partial_\lambda \left( \partial_\kappa A_m - \partial_m A_\kappa \right) + \partial_\mu A_\nu - \partial_\nu A_\mu \]

\[ = -\frac{1}{16\pi} \left[ (\delta_\lambda^\nu \delta_\kappa^\mu - \delta_\kappa^\nu \delta_\lambda^\mu) \partial_\lambda A_m + \partial_\lambda \left( \delta_\lambda^\nu \partial_m A_\kappa - \partial_m A_\kappa \right) \right] \]

\[ = -\frac{1}{16\pi} \left[ -F_{\mu\nu} + F_{\nu\mu} - F_{\lambda\nu} - F_{\lambda\mu} \right] \]

\[ \Rightarrow F_{\mu\nu} = -F_{\nu\mu} \]

\[ \Rightarrow \frac{\partial F_{\mu\nu}}{\partial A_m} = \frac{1}{16\pi} F_{\mu\nu} \]

Adding: \[ \frac{\partial}{\partial A_m} - \partial_\mu \left( \frac{\partial F^{\mu\nu}}{\partial A_m} \right) = 0 \Rightarrow -\frac{1}{3} \tilde{A}^m = \frac{1}{16\pi} \partial_\mu F^{\mu\nu} = 0 = -\frac{1}{3} \tilde{A}^m + \frac{1}{16\pi} \partial_\mu F^{\mu\nu} \]

\[ \Rightarrow \partial_\mu F^{\mu\nu} = \frac{1}{16\pi} \tilde{A}^m \]

The best way to understand the lengthy procedure above is as follows.

Suppose we start w/ \( \mathcal{L} = -\frac{1}{16\pi} \varepsilon_{\mu\nu\lambda\kappa} \partial_\lambda F^{\kappa\nu} \).

If we were taking the derivative w.r.t. \( \partial_\mu (A_m) \) we would have to find every place that this term appeared. But since every index in sight is summed over, we should expect \( \partial_\mu A_m \) to appear once in each term when we do the sum over indices.

A simpler example is: \( \mathcal{L} = A_\mu A^\mu \Rightarrow \frac{\partial}{\partial A_\mu} = \frac{\partial}{\partial A_\mu} \left( \varepsilon_{\mu\nu\lambda\kappa} A_\nu A_\lambda A_\kappa \right) = \varepsilon_{\mu\nu\lambda\kappa} A_\nu \partial_\lambda A_\kappa + \varepsilon_{\mu\nu\lambda\kappa} A_\nu A_\lambda \partial_\kappa = 2 \varepsilon_{\mu\nu\lambda\kappa} A_\nu \partial_\kappa \]

For example: \( \frac{\partial}{\partial A_0} = \frac{\partial}{\partial A_0} \left( -A_0 A_\mu + A_\mu A_\nu A_\kappa A_\lambda A_\kappa A_\lambda A_\kappa A_\lambda \right) = -2A_0 \)
5. The Dirac equation: \( \gamma^\mu \partial_\mu \Psi + \frac{i\alpha}{\hbar} \Psi = 0 \)

The Klein-Gordon equation: \( \Box \Psi + \frac{(\alpha^2)}{m^2} \Psi = 0 \)

Let's go... If \( (\gamma^\mu \partial_\mu + \frac{i\alpha}{\hbar}) \Psi = 0 \)
then \( (\gamma^\mu \partial_\mu - \frac{i\alpha}{\hbar}) (\gamma^\mu \partial_\mu + \frac{i\alpha}{\hbar}) \Psi = 0 \)
so \( \gamma^\mu \gamma^\nu \partial_\mu \partial_\nu \Psi + \frac{2i\alpha}{m^2} \gamma^\mu \partial_\mu \Psi - \frac{(\alpha^2)}{m^2} \Psi = 0 \)
adding this line to

\[ \Box \Psi + \frac{(\alpha^2)}{m^2} \Psi = 0 \]

there can be freely switched

\( \Box \Psi + \frac{(\alpha^2)}{m^2} \Psi = 0 \)

or \( \Box \Psi + \frac{(\alpha^2)}{m^2} \Psi = 0 \)  \[ \text{Beam}! \]