Particle Physics HW5

1. Show that at least one of the nonzero momentum solutions presented in class actually solves the Dirac equation. **Hint**: It is easier to do this with the Dirac equation written in momentum space, i.e. $\frac{i}{\hbar}\gamma^\mu P_\mu \psi + \frac{mc}{\hbar}\psi = 0$.
2. To find helicity eigenstates class we cheated and aligned our coordinates so that the z-axis pointed along the momentum. In this problem you will show that helicity eigenstates can be constructed for an arbitrary orientation of the coordinates. Explicitly construct the helicity projection operators $P_\pm = \frac{1}{2}\left(1 \pm \frac{2}{\hbar} S_{\vec{p}}\right)$. Evaluate $P_+\psi^{(1)}$ where $\psi^{(1)}$ is the first **nonzero** momentum solution shown in class. Then explicitly show that the result of $P_+\psi^{(1)}$ is an eigenstate of $S_{\vec{p}}$ with eigenvalue $+\frac{\hbar}{2}$. **Hint:** Once you have constructed the explicit form of $S_{\vec{p}}$, the rest is plug and chug. Remember that each component of $S_{\vec{p}}$ should be weighted by the nonzero momentum in that direction, e.g. the z-component should include a factor of $\frac{p_z}{p}$ where here p is the magnitude of the spatial momentum. Also, remember that we are not dealing with **chirality** here, so γ^5 should not be part of your work!
3. Using the definition of γ^5 (**not** the explicit matrix form), show that $\frac{1}{2}(1 \pm \gamma^5)$ are projection operators. **Hint:** You can use any result that you proved in your last homework set.
4. Show that Dirac Lagrangian for a massive field, expressed in terms of Weyl spinors ψ_+ and ψ_-, takes the form shown in class.
5. I want you to work through another example of gauging an abelian symmetry in order to create an interacting theory. In this case, instead of starting with fermions and the Dirac Lagrangian, I want you to consider the free Klein-Gordon Lagrangian, but take the scalar field $\phi(x^\mu)$ to be complex, i.e. $\phi(x^\mu) = \phi_A(x^\mu) + i\phi_B(x^\mu)$. The only modification needed for the Lagrangian is that in each term one of the fields should be $\phi(x^\mu)$ and the other $\phi^*(x^\mu)$, so that the entire Lagrangian is real. So your starting point should be:
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^*\partial^\mu\phi + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \phi^*\phi$$
For this free Lagrangian, you should be able to identify a global symmetry that is strikingly similar to the example we discussed in class.
a) Identify the global symmetry and verify that this Lagrangian is invariant under it.
b) Promote this to a local symmetry as we did in class. In order to do this you will need to define a new covariant derivative which will require the addition of a new gauge field. Determine the required transformation rule for the new gauge field. **Hints:** It might help to remember just how far the first derivative acts and where exactly you can complex conjugate (remember that ∂_μ is real). I find it easier to remember if we write the Lagrangian as
$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^*\left(\partial^\mu\phi\right) + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \phi^*\phi$$
c) Allow the new gauge field to propagate by adding in the appropriate kinetic term and verify that the new term is also invariant.
d) Well, you know what to do next. Cheers!

6. Consider the Proca equation with zero mass. If we look for plane-wave solutions of the form $A^\nu = Ae^{i\frac{P_\mu x^\mu}{\hbar}}\epsilon^\nu$ where ϵ^ν is a polarization vector and P^μ is the four-momentum, show that the four –momentum and polarization are "orthogonal" in the 4D sense.