1. Dirac: $\gamma^\mu \partial_\mu \psi + \frac{mc}{\hbar}\psi = 0$

Consider $\psi^{(2)} = A e^{\frac{i}{\hbar} p_\mu x^\mu} \overbrace{\begin{pmatrix} -\frac{p_x}{mc} + i\frac{p_y}{mc} \\ \frac{E}{mc^2} + \frac{p_z}{mc} \\ 0 \\ 1 \end{pmatrix}}^{u} \Rightarrow \partial_\mu \psi = \frac{i}{\hbar} p_\mu A e^{\frac{i}{\hbar} p_\mu x^\mu} \begin{pmatrix} -\frac{p_x}{mc} + i\frac{p_y}{mc} \\ \frac{E}{mc^2} + \frac{p_z}{mc} \\ 0 \\ 1 \end{pmatrix} = \frac{i}{\hbar} p_\mu \psi$

Then the Dirac equation becomes $\frac{i}{\hbar}\gamma^\mu p_\mu \psi + \frac{mc}{\hbar}\psi = 0$ or $\underbrace{\gamma^\mu p_\mu \psi = imc\psi}_{\text{The } Ae^{\frac{i}{\hbar}p_\mu x^\mu} \text{ is on both sides so cancel.}} \Rightarrow \gamma^\mu p_\mu u = imc\, u$

Left hand side: $\gamma^\mu p_\mu \psi = (\gamma^0 p_0 + \gamma^1 p_1 + \gamma^2 p_2 + \gamma^3 p_3)\psi$

$$= \left(\gamma^0\left(-\frac{E}{c}\right) + \gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z\right)\psi$$

$$= \left[\begin{pmatrix} 0 & 0 & \frac{iE}{c} & 0 \\ 0 & 0 & 0 & \frac{iE}{c} \\ \frac{iE}{c} & 0 & 0 & 0 \\ 0 & \frac{iE}{c} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -ip_x \\ 0 & 0 & -ip_x & 0 \\ 0 & ip_x & 0 & 0 \\ ip_x & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -p_y \\ 0 & 0 & p_y & 0 \\ 0 & p_y & 0 & 0 \\ -p_y & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -ip_z & 0 \\ 0 & 0 & 0 & ip_z \\ ip_z & 0 & 0 & 0 \\ 0 & -ip_z & 0 & 0 \end{pmatrix}\right] u$$

$$= \begin{pmatrix} 0 & 0 & i\left(\frac{E}{c} - p_z\right) & -ip_x - p_y \\ 0 & 0 & -ip_x + p_y & i\left(\frac{E}{c} + p_z\right) \\ i\left(\frac{E}{c} + p_z\right) & ip_x + p_y & 0 & 0 \\ ip_x - p_y & i\left(\frac{E}{c} - p_z\right) & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{p_x}{mc} + i\frac{p_y}{mc} \\ \frac{E}{mc^2} + \frac{p_z}{mc} \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -ip_x - p_y \\ i\left(\frac{E}{c} + p_z\right) \\ i\left(\frac{E}{c} + p_z\right)\left(-\frac{p_x}{mc} + i\frac{p_y}{mc}\right) + (ip_x + p_y)\left(\frac{E}{mc^2} + \frac{p_z}{mc}\right) \\ (ip_x - p_y)\left(-\frac{p_x}{mc} + i\frac{p_y}{mc}\right) + i\left(\frac{E}{c} - p_z\right)\left(\frac{E}{mc^2} + \frac{p_z}{mc}\right) \end{pmatrix}$$

$$= \begin{pmatrix} -ip_x - p_y \\ i\left(\frac{E}{c} + p_z\right) \\ 0 \\ \underbrace{-\frac{ip_x^2}{mc} - \frac{ip_y^2}{mc} + \frac{iE^2}{mc^3} - \frac{ip_z^2}{mc}}_{\frac{E^2}{c^2} - p^2 = m^2c^2} \end{pmatrix} = \begin{pmatrix} -ip_x - p_y \\ i\left(\frac{E}{c} + p_z\right) \\ 0 \\ imc \end{pmatrix}$$

Right hand side: $imc\, u = \begin{pmatrix} -ip_x - p_y \\ i\left(\frac{E}{c} + p_z\right) \\ 0 \\ imc \end{pmatrix}$ $\checkmark$

2. $P_\pm = \frac{1}{2}\left[1 \pm \frac{2}{\hbar} S_{\vec{p}}\right] = \frac{1}{2}\left[1 \pm \frac{2}{\hbar}\left(\frac{p_x}{p}S_x + \frac{p_y}{p}S_y + \frac{p_z}{p}S_z\right)\right]$

where $S_i = \frac{\hbar}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$ $\quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$P_+\psi^{(1)} = \frac{1}{2}\left[\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \frac{p_x}{p}\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} + \frac{p_y}{p}\begin{pmatrix} 0 & -i & & \\ i & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{pmatrix} + \frac{p_z}{p}\begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & 1 & 0 \\ & & 0 & -1 \end{pmatrix}\right]\psi^{(1)}$$

$$= \frac{1}{2}\,\frac{1}{p}\begin{bmatrix} p+p_z & p_x - ip_y & & \\ p_x + ip_y & p - p_z & & \\ & & p+p_z & p_x - ip_y \\ & & p_x + ip_y & p - p_z \end{bmatrix} A e^{\frac{i p_\mu x^\mu}{\hbar}} \frac{1}{mc}\begin{pmatrix} \frac{E}{c} - p_z \\ -p_x - ip_y \\ mc \\ 0 \end{pmatrix}$$

Note: I factored out an overall $\frac{1}{p}$ from the first and mc from the second!

$$= \frac{1}{2} A e^{\frac{i p_\mu x^\mu}{\hbar}} \frac{1}{pmc}\begin{bmatrix} (p+p_z)(\frac{E}{c} - p_z) - (p_x - ip_y)(p_x + ip_y) \\ (p_x + ip_y)(\frac{E}{c} - p_z) - (p - p_z)(p_x + ip_y) \\ (p+p_z)mc \\ (p_x + ip_y)mc \end{bmatrix}$$

$$= \frac{1}{2} A e^{\frac{i p_\mu x^\mu}{\hbar}} \frac{1}{pmc}\begin{bmatrix} \frac{pE}{c} - pp_z + \frac{p_z E}{c} - p_z^2 - p_x^2 - p_y^2 \\ \frac{p_x E}{c} + \frac{ip_y E}{c} - p_x p_z - ip_y p_z - pp_x - ipp_y + p_z p_x + ip_z p_y \\ (p+p_z)mc \\ (p_x + ip_y)mc \end{bmatrix}$$

Recall: $p^2 = p_x^2 + p_y^2 + p_z^2$

$$= \frac{1}{2} A e^{\frac{i p_\mu x^\mu}{\hbar}} \frac{1}{pmc}\begin{bmatrix} \frac{pE}{c} + \frac{p_z E}{c} - p^2 - pp_z \\ \frac{p_x E}{c} + \frac{ip_y E}{c} - pp_x - ipp_y \\ (p+p_z)mc \\ (p_x + ip_y)mc \end{bmatrix} = \frac{1}{2} A e^{\frac{i p_\mu x^\mu}{\hbar}} \frac{1}{pmc}\begin{bmatrix} (\frac{E}{c} - p)(p+p_z) \\ (\frac{E}{c} - p)(p_x + ip_y) \\ (p+p_z)mc \\ (p_x + ip_y)mc \end{bmatrix}$$

We have $P_+ \psi^{(1)} = \frac{1}{2} A e^{i\frac{p_\mu x^\mu}{\hbar}} \frac{1}{pmc} \begin{bmatrix} (\frac{E}{c} - p)(p + p_z) \\ (\frac{E}{c} - p)(p_x + i p_y) \\ (p + p_z) mc \\ (p_x + i p_y) mc \end{bmatrix} \equiv \psi_+^{(1)}$

To verify this as an eigenstate of $S_{\vec{p}}$ explicitly we apply $S_{\vec{p}} = \frac{\hbar}{2}\left[\frac{p_x}{p} S_x + \frac{p_y}{p} S_y + \frac{p_z}{p} S_z\right]$

$$= \frac{\hbar}{2p}\left[p_x S_x + p_y S_y + p_z S_z\right]$$

$$= \frac{\hbar}{2p}\begin{bmatrix} p_z & p_x - i p_y & & \\ p_x + i p_y & -p_z & & \\ & & p_z & p_x - i p_y \\ & & p_x + i p_y & -p_z \end{bmatrix}$$

$$S_{\vec{p}} \psi_+^{(1)} = \frac{\hbar}{2p} \frac{1}{2} A e^{i\frac{p_\mu x^\mu}{\hbar}} \frac{1}{pmc} \begin{bmatrix} p_z & p_x - i p_y & & \\ p_x + i p_y & -p_z & & \\ & & p_z & p_x - i p_y \\ & & p_x + i p_y & -p_z \end{bmatrix} \begin{bmatrix} (\frac{E}{c} - p)(p + p_z) \\ (\frac{E}{c} - p)(p_x + i p_y) \\ (p + p_z) mc \\ (p_x + i p_y) mc \end{bmatrix}$$

$$= \frac{\hbar}{2p} \frac{1}{2} A e^{i\frac{p_\mu x^\mu}{\hbar}} \frac{1}{pmc} \begin{bmatrix} p_z (\frac{E}{c} - p)(p + p_z) + (p_x - i p_y)(\frac{E}{c} - p)(p_x + i p_y) \\ (p_x + i p_y)(\frac{E}{c} - p)(p + p_z) - p_z(\frac{E}{c} - p)(p_x + i p_y) \\ p_z (p + p_z) mc + (p_x - i p_y)(p_x + i p_y) mc \\ (p_x + i p_y)(p + p_z) mc - p_z (p_x + i p_y) mc \end{bmatrix}$$

$$= \frac{\hbar}{2p} \frac{1}{2} A e^{i\frac{p_\mu x^\mu}{\hbar}} \frac{1}{pmc} \begin{bmatrix} (\frac{E}{c} - p)(p_z p + p_z^2 + p_x^2 + p_y^2) \\ (\frac{E}{c} - p)(p_x p + p_x p_z + i p_y p + i p_y p_z - p_z p_x - i p_z p_y) \\ (p_z p + p_z^2 + p_x^2 + p_y^2) mc \\ (p_x p + p_x p_z + i p_y p + i p_y p_z - p_z p_x - i p_z p_y) mc \end{bmatrix}$$

$$= \frac{\hbar}{2p} \frac{1}{2} A e^{i\frac{p_\mu x^\mu}{\hbar}} \frac{1}{pmc} \begin{bmatrix} (\frac{E}{c} - p)(p_z + p) p \\ (\frac{E}{c} - p)(p_x + i p_y) p \\ (p_z + p) mc\, p \\ (p_x + i p_y) mc\, p \end{bmatrix}$$

cancel

$$= \frac{\hbar}{2} \frac{1}{2} A e^{i\frac{p_\mu x^\mu}{\hbar}} \frac{1}{pmc} \begin{bmatrix} (\frac{E}{c} - p)(p + p_z) \\ (\frac{E}{c} - p)(p_x + i p_y) \\ (p + p_z) mc \\ (p_x + i p_y) mc \end{bmatrix} = +\frac{\hbar}{2} \psi_+^{(1)} \quad \checkmark$$

3. $P_\pm = \frac{1}{2}(1 \pm \gamma^5)$

First of all, any projection operator must be idempotent, that is $P^2 = P$.

To show this:

$P_+ P_+ = \frac{1}{4}(1+\gamma^5)(1+\gamma^5) = \frac{1}{4}(1+\gamma^5+\gamma^5+\gamma^5\gamma^5)$ but recall that $\gamma^5\gamma^5 = 1$ so

$= \frac{1}{4}(2+2\gamma^5) = \frac{1}{2}(1+\gamma^5) = P_+$

$P_- P_- = \frac{1}{4}(1-\gamma^5)(1-\gamma^5) = \frac{1}{4}(1-\gamma^5-\gamma^5+\gamma^5\gamma^5) = \frac{1}{4}(2-2\gamma^5) = P_-$

Also we can show that they project to different subspaces via $P_+ P_- = P_- P_+ = 0$

$P_+ P_- = \frac{1}{4}(1+\gamma^5)(1-\gamma^5) = \frac{1}{4}(1+\gamma^5-\gamma^5-\gamma^5\gamma^5) = 0$

4. $\mathcal{L}_{Dirac} = \hbar c \bar{\psi} \gamma^\mu \partial_\mu \psi + mc^2 \bar{\psi}\psi$

For $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \Rightarrow \bar{\psi} = (\psi_-^+ \ \psi_+^+)$ from class.

$\gamma^0 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = -i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \Rightarrow \quad \gamma^\mu = -i \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ where $\sigma^\mu = 1 + \sigma^i$, $\bar{\sigma}^\mu = 1 - \sigma^i$

Then: $\mathcal{L}_{Dirac} = \hbar c (\psi_-^+ \ \psi_+^+)(-i) \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} \partial_\mu \psi_+ \\ \partial_\mu \psi_- \end{pmatrix} + mc^2 (\psi_-^+ \psi_+^+) \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$

$= -i\hbar c (\psi_-^+ \ \psi_+^+) \begin{pmatrix} \sigma^\mu \partial_\mu \psi_- \\ \bar{\sigma}^\mu \partial_\mu \psi_+ \end{pmatrix} + mc^2 (\psi_-^+ \psi_+ + \psi_+^+ \psi_-)$

$= -i\hbar c \, \psi_-^+ \sigma^\mu \partial_\mu \psi_- - i\hbar c \, \psi_+^+ \bar{\sigma}^\mu \partial_\mu \psi_+ + mc^2 (\psi_-^+ \psi_+ + \psi_+^+ \psi_-)$ ✓

5. $\mathcal{L} = \frac{1}{2}\partial_\mu \phi^* \partial^\mu \phi + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \phi^* \phi$

a) Consider $\phi \to \phi' = e^{iq\theta}\phi \Rightarrow \phi^* \to \phi^{*\prime} = \phi^* e^{-iq\theta}$ for θ = constant

Then $\mathcal{L} \to \mathcal{L}' = \frac{1}{2}\partial_\mu \phi^* e^{-iq\theta}\partial^\mu e^{iq\theta}\phi + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \phi^* \underbrace{e^{-iq\theta} e^{iq\theta}}_{1} \phi$

$= \mathcal{L}$

b) Now consider $\phi \to \phi' = e^{iq\theta(x^\nu)}\phi \Rightarrow \phi^* \to \phi^{*\prime} = \phi^* e^{-iq\theta(x^\nu)}$ w/ $D_\mu \equiv \partial_\mu + iqA_\mu(x^\nu)$

These parentheses are very important! and $A_\mu \to A_\mu'$ to be determined

$\mathcal{L} = \frac{1}{2}(D_\mu\phi)^* D^\mu \phi + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \phi^*\phi = \frac{1}{2}(\partial_\mu + iqA_\mu)\phi^*(\partial^\mu + iqA^\mu)\phi + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^*\phi$

$\mathcal{L} \to \mathcal{L}' = \frac{1}{2}(\partial_\mu - iqA_\mu')\phi^* e^{-iq\theta(x^\nu)}(\partial^\mu + iqA^{\mu\prime})e^{iq\theta(x^\nu)}\phi + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \phi^* \underbrace{e^{-iq\theta(x^\nu)} e^{iq\theta(x^\nu)}}_{1}\phi$

$\frac{1}{2}\eta^{\mu\nu}\left[\underbrace{(\partial_\mu - iqA_\mu')\phi^* e^{-iq\theta(x^\nu)}}\right]\left[\underbrace{(\partial_\nu + iqA_\nu')e^{iq\theta(x^\nu)}\phi}\right]$

Nothing here acts on anything here!

We won't worry about this term since it is already invariant.

What we would like to achieve is to have $\begin{cases} D_\mu^{*\prime} e^{-iq\theta(x^\nu)}\phi^* = e^{-iq\theta(x^\nu)} D_\mu \phi^* \\ D_\nu' e^{iq\theta(x^\nu)}\phi = e^{iq\theta(x^\nu)} D_\nu \phi \end{cases}$ Since then we can cancel exponents!

Working w/ one of these: $(\partial_\nu + iqA_\nu')e^{iq\theta(x^\nu)}\phi = iq(\partial_\nu\theta(x^\nu))e^{iq\theta(x^\nu)}\phi + e^{iq\theta(x^\nu)}\partial_\nu\phi + iqA_\nu' e^{iq\theta(x^\nu)}\phi$

We want this equal to $e^{iq\theta(x^\nu)}(\partial_\nu + iqA_\nu)\phi$ which will be the case if $A_\nu' = A_\nu - \partial_\nu\theta(x^\nu)$

But this also fixes: $(\partial_\mu - iqA_\mu')e^{-iq\theta(x^\nu)}\phi^* = -iq(\partial_\mu\theta(x^\nu))e^{-iq\theta(x^\nu)}\phi^* + e^{-iq\theta(x^\nu)}\partial_\mu\phi^* - iqA_\mu' e^{-iq\theta(x^\nu)}\phi^*$

So $\mathcal{L} = \frac{1}{2}(D_\mu\phi)^* D^\mu\phi + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^*\phi$ is invariant under $\phi \to \phi' = e^{iq\theta(x^\nu)}\phi$
w/ $D_\mu = \partial_\mu + iqA_\mu$ $\quad \phi^* \to \phi^{*\prime} = \phi^* e^{-iq\theta(x^\nu)}$
$A_\mu \to A_\mu' = A_\mu - \partial_\mu\theta(x^\nu)$

c) Since A_μ is a (dual)-vector field and the gauge transformation is the same as in the Dirac case we did in class, we know that to allow A_μ to propagate we should just add the Proca Lagrangian w/ $m_{A_\mu} = 0$ (for gauge invariance).

$\mathcal{L} = \frac{1}{2}(D_\mu\phi)^* D^\mu\phi + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^*\phi + \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$ w/ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

d) Cheers!

6. $\partial_\mu F^{\mu\nu} = 0 \qquad A^\nu = A e^{\frac{i}{\hbar} p_\mu x^\mu} \epsilon^\nu \Rightarrow \partial_\mu A^\nu = \frac{i}{\hbar} p_\mu A e^{\frac{i}{\hbar} p_\mu x^\mu} \epsilon^\nu = \frac{i}{\hbar} p_\mu A^\nu$

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\mu(\eta^{\mu\lambda}\partial_\lambda A^\nu - \eta^{\nu\rho}\partial_\rho A^\mu) = \partial_\mu\left(\eta^{\mu\lambda}\frac{i}{\hbar}p_\lambda A e^{\frac{i}{\hbar}p_\lambda x^\lambda}\epsilon^\nu - \eta^{\nu\rho}p_\rho A e^{\frac{i}{\hbar}p_\rho x^\rho}\epsilon^\mu\right)$$

$$= \left(\frac{i}{\hbar}\right)^2\left[\eta^{\mu\lambda}p_\lambda p_\mu A e^{\frac{i}{\hbar}p_\mu x^\mu}\epsilon^\nu - \eta^{\nu\rho}p_\rho p_\mu A e^{\frac{i}{\hbar}p_\mu x^\mu}\epsilon^\mu\right] = 0$$

$$\Rightarrow \eta^{\mu\lambda}p_\lambda p_\mu \epsilon^\nu - \eta^{\nu\rho}p_\rho p_\mu \epsilon^\mu = 0$$

$$\underbrace{p^\mu p_\mu}_{= -m^2c^2 = 0 \text{ for } m=0 \Rightarrow} \epsilon^\nu - p^\nu \underbrace{p_\mu \epsilon^\mu}_{=0 \text{ since } p^\nu \neq 0} = 0$$

Thus $p_\mu \epsilon^\mu = 0$ which is orthogonality in 4D.