Particle Physics HW6

1. Starting with a theory that is invariant under a local non-abelian gauge symmetry like $SU(3)$ for QCD, show that it reduces to electromagnetism when the symmetry is taken to be abelian instead. There are two important things to check: 1) The transformation of the gauge field and 2) The kinetic term for the gauge field. **This should not be a long problem. I am just trying to get you to look at and compare the gauging result for abelian vs. non-abelian symmetries.**
2. I want you to work through an example of gauging a non-abelian symmetry in order to create an interacting theory. In this case, instead of starting with fermions and the Dirac Lagrangian, I want you to consider the free Klein-Gordon Lagrangian, but take the scalar field $\phi(x^\mu)$ to be a real three component object, i.e. $\phi(x^\mu) = (\phi_A(x^\mu), \phi_B(x^\mu), \phi_C(x^\mu))$. This time the Lagrangian will utilize the transpose of the field $\phi^T(x^\mu)$. So your starting point should be:
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^T\partial^\mu\phi + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \phi^T\phi$$
For this free Lagrangian, you should be able to identify a non-abelian global symmetry.
a) Identify the global symmetry and verify that this Lagrangian is invariant under it.
b) Promote this to a local symmetry as we did in class. In order to do this you will need to define a new covariant derivative which will require the addition of a new gauge field. Determine the required transformation rule for the new gauge field.
c) Allow the new gauge field to propagate by adding in the appropriate kinetic term. You do not need to verify that the new term is gauge invariant (although it is).
d) Well, you know what to do next. Cheers!
3. Consider a theory of **both** left and right-handed matter doublets $\chi_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$ and $\chi_R = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_R$ which enjoys a global $SU(2)_L \times SU(2)_R$ invariance.
a) Write down an appropriate Lagrangian for these fields with this global symmetry. Can the matter fields have mass in this case? Include a mass term if it is allowed.
b) Gauge it as usual.
c) Let the gauge field wiggle about.
d) Prost!
4. As a follow up from the gauging story, I now want you to demonstrate that the non-abelian gauge field kinetic term $\frac{1}{16\pi}F^a_{\mu\nu}F^{a\,\mu\nu}$ is itself gauge invariant. To do so it is easier to call the transformations $e^{ig\lambda\cdot\theta(x)}$ which are in general a set of non-commuting matrices (since the λ are) by $U(x)$ and just keep in mind that $U(x)$ are non-commuting matrices. Also, you can work with the full set of gauge fields all at once by using $A_\mu \equiv \lambda^a A^a_\mu$ and hence work in terms of $F_{\mu\nu} \equiv \lambda^a F^a_{\mu\nu}$, again as long as you keep in mind that A_μ and $F_{\mu\nu}$ are non-commuting objects.
   a) Write the gauge field transformation law for A_μ in terms of the matrices $U(x)$.
   b) Now determine how the gauge field strength $F_{\mu\nu}$ will transform in terms of $U(x)$. **Hint:** You will need to use the following (which you should prove), $\partial_\mu(U^{-1}) = -U^{-1}\partial_\mu(U)U^{-1}$.
   c) Finally consider $F^a_{\mu\nu}F^{a\,\mu\nu}$ which in this language is just $Tr(F_{\mu\nu}F^{\mu\nu})$ where the trace is over the "color" space matrices. Demonstrate that this combination is invariant. It will help to

recall that spacetime indices being upper or lower does not change anything with regards to gauge transformations, i.e. $F_{\mu\nu}$ and $F^{\mu\nu}$ will transform the same way.

d) Going back to your result from part (b), verify that if the group is actually abelian, then $F_{\mu\nu}$ is invariant on its own.