1. Consider the Lagrangian for QCD based on $SU(3)$:

$$\mathcal{L} = \hbar c\, \bar{\psi} \gamma^\mu (\partial_\mu + i g \lambda \cdot A_\mu)\psi + mc^2 \bar{\psi}\psi + \frac{1}{16\pi}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} - g f^{ade} A^{\mu d} A^{\nu e})$$

Which is invariant under:

$$\psi \to \psi' = e^{-i \frac{g}{\hbar c} \lambda \cdot \phi}\psi, \quad \bar{\psi} \to \bar{\psi}' = \bar{\psi} e^{i \frac{g}{\hbar c}\lambda \cdot \phi}, \qquad \lambda \cdot A_\mu \to \lambda \cdot A_\mu' = e^{-ig\lambda\cdot\phi}\lambda \cdot A_\mu e^{ig\lambda\cdot\phi} + \frac{i}{g}\partial_\mu(e^{-ig\lambda\cdot\phi})e^{ig\lambda\cdot\phi}$$

For local transformation parameters $\phi(x)$.

If we instead restrict to an abelian group then the Lie algebra of generators becomes $[\lambda^a, \lambda^b] = 0 \Rightarrow f^{abc} = 0$

We can immediately see that the gauge field kinetic term reduces to $\frac{1}{16\pi}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a})$ which is what we expect for an abelian symmetry, e.g. $U(1)$ of E&M (or even something like $U(1)\times U(1)$).

As for the gauge field transformation, since all of the λ's commute, there is no problem moving the exponentials around, i.e.

$$\lambda \cdot A_\mu' = e^{-ig\lambda\cdot\phi}\lambda \cdot A_\mu e^{ig\lambda\cdot\phi} + \frac{i}{g}\partial_\mu(e^{-ig\lambda\cdot\phi})e^{ig\lambda\cdot\phi}$$

This is okay if all the λ's commute.

Normally you would have to be very careful taking this derivative. For example would you write $\partial_\mu(-ig\lambda\cdot\phi)e^{-ig\lambda\cdot\phi}$ or $e^{-ig\lambda\cdot\phi}\partial_\mu(-ig\lambda\cdot\phi)$?
Fortunately, for the abelian case it does not matter since they are the same!

Then $\lambda \cdot A_\mu' = \lambda \cdot A_\mu + \frac{i}{g}\partial_\mu(-ig\lambda\cdot\phi)$ having cancelled all the exponentials

$$= \lambda \cdot A_\mu + \lambda \cdot \partial_\mu \phi$$

If we only have one generator this becomes: $A_\mu' = A_\mu + \partial_\mu \phi$ as expected.

2. $\mathcal{L} = \frac{1}{2}\partial_\mu \phi^T \partial^\mu \phi + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \phi^T \phi$ where $\phi = \begin{pmatrix} \phi_A \\ \phi_B \\ \phi_C \end{pmatrix}$

a) This Lagrangian is invariant under a global symmetry that transforms $\phi \to \phi' = M\phi$, $\phi^T \to \phi'^T = (M\phi)^T = \phi^T M^T$ as long as $\phi^T\phi \to \phi^T M^T M \phi = \phi^T \phi$ or $M^T M = I$. Since M is 3×3 and real this is $SO(3)$.
Note: $M^T M = I$ defines $O(N)$, but we also need $\det M = +1$ so that this is a Lie group.
As usual we can write $M = e^{ig\lambda\cdot\theta}$ where $\lambda = (\lambda^1, \lambda^2, \lambda^3)$ are the generators of $SO(3)$ w/ $[\lambda^i, \lambda^j] = i\epsilon^{ijk}\lambda^k$ and $\theta = (\theta^1, \theta^2, \theta^3)$ is a vector of parameters

b) $\partial_\mu \phi \to D_\mu \phi = \partial_\mu \phi + ig\lambda\cdot A_\mu \phi$

For invariance we need $D_\mu \phi \to D'_\mu \phi' = e^{ig\lambda\cdot\theta(x^\nu)} D_\mu \phi$

Then: $D'_\mu \phi' = \partial_\mu \phi' + ig\lambda\cdot A'_\mu \phi' = \partial_\mu (e^{ig\lambda\cdot\theta}\phi) + ig\lambda\cdot A'_\mu e^{ig\lambda\cdot\theta}\phi$

$$= \partial_\mu (e^{ig\lambda\cdot\theta})\phi + e^{ig\lambda\cdot\theta}\partial_\mu\phi + ig\lambda\cdot A'_\mu e^{ig\lambda\cdot\theta}\phi$$

we want $= e^{ig\lambda\cdot\theta}\left[\partial_\mu\phi + ig\lambda\cdot A_\mu\phi\right]$

which we get if $\lambda\cdot A'_\mu = e^{ig\lambda\cdot\theta}\lambda\cdot A_\mu e^{-ig\lambda\cdot\theta} + \frac{i}{g}\partial_\mu(e^{ig\lambda\cdot\theta})e^{-ig\lambda\cdot\theta}$

Now the derivative term in $\mathcal{L}$ should be understood as $\frac{1}{2}(D^\mu\phi)^T D_\mu\phi$ and we now have:

$$\frac{1}{2}(D^\mu\phi)^T D_\mu\phi \longrightarrow \frac{1}{2}(D'^\mu\phi')^T D'_\mu\phi' = \frac{1}{2}(e^{ig\lambda\cdot\theta}D^\mu\phi)^T e^{ig\lambda\cdot\theta}D_\mu\phi = \frac{1}{2}(D^\mu\phi)^T\underbrace{(e^{ig\lambda\cdot\theta})^T}_{\text{the only matrix here is }\lambda}e^{ig\lambda\cdot\theta}D_\mu\phi$$

But for $SO(3)$ the generators satisfy $\lambda^T = -\lambda$ so this gives $\frac{1}{2}(D^\mu\phi)^T e^{-ig\lambda\cdot\theta}e^{ig\lambda\cdot\theta}D_\mu\phi = \frac{1}{2}(D^\mu\phi)^T D_\mu\phi$

c) $F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu] = -\frac{i}{g}\left[\partial_\mu + ig\lambda\cdot A_\mu, \partial_\nu + ig\lambda\cdot A_\nu\right]\phi$

$$= -\frac{i}{g}\Big[\cancel{\partial_\mu\partial_\nu\phi} + \partial_\mu(ig\lambda\cdot A_\nu\phi) + ig\lambda\cdot A_\mu\partial_\nu\phi - g^2\lambda\cdot A_\mu\lambda\cdot A_\nu\phi$$
$$- \cancel{\partial_\nu\partial_\mu\phi} - \partial_\nu(ig\lambda\cdot A_\mu\phi) - ig\lambda\cdot A_\nu\partial_\mu\phi + g^2\lambda\cdot A_\nu\lambda\cdot A_\mu\phi\Big]$$

$$= -\frac{i}{g}\Big[ig\partial_\mu(\lambda\cdot A_\nu)\phi + \cancel{ig\lambda\cdot A_\nu\partial_\mu\phi} + \cancel{ig\lambda\cdot A_\mu\partial_\nu\phi} - g^2\lambda\cdot A_\mu\lambda\cdot A_\nu\phi$$
$$- ig\partial_\nu(\lambda\cdot A_\mu)\phi - \cancel{ig\lambda\cdot A_\mu\partial_\nu\phi} - \cancel{ig\lambda\cdot A_\nu\partial_\mu\phi} + g^2\lambda\cdot A_\nu\lambda\cdot A_\mu\phi\Big]$$

$= \partial_\mu(\lambda\cdot A_\nu) - \partial_\nu(\lambda\cdot A_\mu) + ig\left[\lambda\cdot A_\mu, \lambda\cdot A_\nu\right]$ Dropping overall ϕ

$$= \lambda\cdot\partial_\mu A_\nu - \lambda\cdot\partial_\nu A_\mu + ig\left[\lambda^i, \lambda^j\right]A^i_\mu A^j_\nu$$

$$= \lambda\cdot\partial_\mu A_\nu - \lambda\cdot\partial_\nu A_\mu + ig\,\epsilon^{ijk}A^i_\mu A^j_\nu\lambda^k$$

$$= \lambda^k\left(\partial_\mu A^k_\nu - \partial_\nu A^k_\mu + ig\,\epsilon^{ijk}A^i_\mu A^j_\nu\right) = \lambda^k F^k_{\mu\nu}$$

Then adding $\frac{1}{16\pi}\underbrace{F^k_{\mu\nu}F^{\mu\nu k}}_{\text{sum over }k}$ to $\mathcal{L}$ will allow A_μ to propate.

d)

3. a) $$\mathcal{L} \overset{?}{=} \underbrace{\hbar c\, \bar{\chi}_R \gamma^\mu \partial_\mu \chi_R + \hbar c\, \bar{\chi}_L \gamma^\mu \partial_\mu \chi_L}_{\text{terms mixing L and R vanish due to } \gamma^\mu} + \underbrace{mc^2\, \bar{\chi}_R \chi_L + mc^2\, \bar{\chi}_L \chi_R}_{\text{terms with same RR or LL vanish automatically}}$$

__BUT__: Since $SU(2)_L \times SU(2)_R$ allows independent transformations of χ_L and χ_R, there is no way the mass terms are invariant under all elements of $SU(2)_L \times SU(2)_R$.

So we start w/:

$$\mathcal{L} = \hbar c\, \bar{\chi}_R \gamma^\mu \partial_\mu \chi_R + \hbar c\, \bar{\chi}_L \gamma^\mu \partial_\mu \chi_L$$

where $SU(2)_L$ acts on χ_L as $e^{-ig_L \vec{\sigma}\cdot\vec{\theta}_L} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$ and $SU(2)_R$ acts on χ_R as $e^{-ig_R \vec{\sigma}\cdot\vec{\theta}_R} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_R$

Notice that $g_L \neq g_R$ since these are two different groups, $\vec{\sigma}$ is the same in each case since each is $SU(2)$ and hence the have the same generators, and finally $\vec{\theta}_L \neq \vec{\theta}_R$ since we can do independent transformations in each factor of $SU(2)_L$ and $SU(2)_R$.

b) To gauge we replace $\partial_\mu \chi_L \Rightarrow D_\mu \chi_L = \partial_\mu \chi_L + i g_L \vec{\sigma}\cdot\vec{W}_{L\mu} \chi_L$

$\partial_\mu \chi_R \Rightarrow D_\mu \chi_R = \partial_\mu \chi_R + i g_R \vec{\sigma}\cdot\vec{W}_{R\mu} \chi_R$

Note: 6 new gauge fields, 3 $\vec{W}_{L\mu}$ and 3 $\vec{W}_{R\mu}$

To have $D_\mu \chi_L \to D'_\mu \chi'_L = e^{-ig_L \vec{\sigma}\cdot\vec{\theta}_L} D_\mu \chi_L$ we need:

$$D'_\mu \chi'_L = \partial_\mu \chi'_L + i g_L \vec{\sigma}\cdot\vec{W}'_{L\mu} \chi'_L = \partial_\mu (e^{-ig_L \vec{\sigma}\cdot\vec{\theta}_L} \chi_L) + i g_L \vec{\sigma}\cdot\vec{W}'_{L\mu} e^{-ig_L \vec{\sigma}\cdot\vec{\theta}_L} \chi_L$$

$$= \partial_\mu (e^{-ig_L \vec{\sigma}\cdot\vec{\theta}_L}) \chi_L + e^{-ig_L \vec{\sigma}\cdot\vec{\theta}_L} \partial_\mu \chi_L + i g_L \vec{\sigma}\cdot\vec{W}'_{L\mu} e^{-ig_L \vec{\sigma}\cdot\vec{\theta}_L} \chi_L$$

which we want to be $= e^{-ig_L \vec{\sigma}\cdot\vec{\theta}_L} \partial_\mu \chi_L + e^{-ig_L \vec{\sigma}\cdot\vec{\theta}_L} i g_L \vec{\sigma}\cdot\vec{W}_{L\mu} \chi_L$

Thus we need $\vec{\sigma}\cdot\vec{W}'_{L\mu} = e^{-ig_L \vec{\sigma}\cdot\vec{\theta}_L} \vec{\sigma}\cdot\vec{W}_{L\mu} e^{ig_L \vec{\sigma}\cdot\vec{\theta}_L} + \frac{i}{g} \partial_\mu (e^{-ig_L \vec{\sigma}\cdot\vec{\theta}_L}) e^{ig_L \vec{\sigma}\cdot\vec{\theta}_L}$

and similarly for $\vec{\sigma}\cdot\vec{W}'_{R\mu}$.

c) To allow both sets of gauge fields to propagate we add gauge kinetic terms of the form:

$$-\frac{1}{16\pi} F^a_{L\mu\nu} F^{\mu\nu\, a}_L - \frac{1}{16\pi} F^j_{R\mu\nu} F^{\mu\nu\, j}_R$$

where $F^a_{L\mu\nu} = \partial_\mu W^a_{L\nu} - \partial_\nu W^a_{L\mu} - g\epsilon^{abc} W^b_{L\mu} W^c_{L\nu}$

and similarly for $F^j_{R\mu\nu}$.

d) 

4. a) $A_\mu \to A_\mu' = U A_\mu U^{-1} + \frac{i}{g}\partial_\mu(U)U^{-1}$

b) $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \to F_{\mu\nu}'$

$$F_{\mu\nu}' = \partial_\mu\left(UA_\nu U^{-1} + \frac{i}{g}\partial_\nu(U)U^{-1}\right) - \partial_\nu\left(UA_\mu U^{-1} + \frac{i}{g}\partial_\mu(U)U^{-1}\right)$$
$$+ ig\left[UA_\mu U^{-1} + \frac{i}{g}\partial_\mu(U)U^{-1},\ UA_\nu U^{-1} + \frac{i}{g}\partial_\nu(U)U^{-1}\right]$$

$$= \cancel{\partial_\mu(U)A_\nu U^{-1}} + U\partial_\mu(A_\nu)U^{-1} + UA_\nu\partial_\mu U^{-1} + \cancel{\frac{i}{g}\partial_\mu\partial_\nu(U)U^{-1}} + \frac{i}{g}\partial_\nu(U)\partial_\mu(U^{-1})$$
$$- \cancel{\partial_\nu(U)A_\mu U^{-1}} - U\partial_\nu(A_\mu)U^{-1} - UA_\mu\partial_\nu U^{-1} - \cancel{\frac{i}{g}\partial_\nu\partial_\mu(U)U^{-1}} - \frac{i}{g}\partial_\mu(U)\partial_\nu(U^{-1})$$
$$+ ig\, UA_\mu A_\nu U^{-1} - UA_\mu U^{-1}\partial_\nu(U)U^{-1} - \cancel{\partial_\mu(U)A_\nu U^{-1}} - \frac{i}{g}\partial_\mu(U)U^{-1}\partial_\nu(U)U^{-1}$$
$$- ig\, UA_\nu A_\mu U^{-1} + UA_\nu U^{-1}\partial_\mu(U)U^{-1} + \cancel{\partial_\nu(U)A_\mu U^{-1}} + \frac{i}{g}\partial_\nu(U)U^{-1}\partial_\mu(U)U^{-1}$$

Using that $\partial_\mu(U^{-1}) = -U^{-1}\partial_\mu(U)U^{-1}$ (replace $U^{-1}\partial(U)U^{-1}$ w/ $-\partial(U^{-1})$ in last 2 lines):

proof: $\partial_\mu(U^{-1}) = \partial_\mu(U^{-1}UU^{-1}) = \partial_\mu(U^{-1})UU^{-1} + U^{-1}\partial_\mu(U)U^{-1} + U^{-1}U\partial_\mu(U^{-1})$
$\partial_\mu(U^{-1}) = \partial_\mu(U^{-1}) + U^{-1}\partial_\mu(U)U^{-1} + \partial_\mu(U^{-1})$
$\Rightarrow \partial_\mu(U^{-1}) = -U^{-1}\partial_\mu(U)U^{-1}$

$$F_{\mu\nu}' = U\partial_\mu(A_\nu)U^{-1} + \cancel{UA_\nu\partial_\mu(U^{-1})} + \cancel{\frac{i}{g}\partial_\nu(U)\partial_\mu(U^{-1})}$$
$$- U\partial_\nu(A_\mu)U^{-1} - \cancel{UA_\mu\partial_\nu(U^{-1})} - \cancel{\frac{i}{g}\partial_\mu(U)\partial_\nu(U^{-1})}$$
$$+ ig\,UA_\mu A_\nu U^{-1} + \cancel{UA_\mu\partial_\nu(U^{-1})} + \cancel{\frac{i}{g}\partial_\mu(U)\partial_\nu(U^{-1})}$$
$$- ig\,UA_\nu A_\mu U^{-1} - \cancel{UA_\nu\partial_\mu(U^{-1})} - \cancel{\frac{i}{g}\partial_\nu(U)\partial_\mu(U^{-1})}$$
$$= U\left\{\partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]\right\}U^{-1} = UF_{\mu\nu}U^{-1}$$

c) $Tr(F_{\mu\nu}F^{\mu\nu}) \to Tr(F_{\mu\nu}'F'^{\mu\nu}) = Tr(UF_{\mu\nu}U^{-1}UF^{\mu\nu}U^{-1}) = Tr(UF_{\mu\nu}F^{\mu\nu}U^{-1})$

$= Tr(U^{-1}UF_{\mu\nu}F^{\mu\nu})$ using cyclicity of Tr

$= Tr(F_{\mu\nu}F^{\mu\nu})$

d) For abelian groups, the U matrices commute w/ everything so $F_{\mu\nu}' = UF_{\mu\nu}U^{-1} = UU^{-1}F_{\mu\nu} = F_{\mu\nu}$