1. $\phi_1 = \frac{1}{\sqrt{2}}\frac{\mu}{\lambda}$, $\phi_2 = \frac{1}{\sqrt{2}}\frac{\mu}{\lambda}$, $A_\mu = 0$ First we will do this by brute force. See next page for trick!

$$\left.\begin{aligned} \phi_1 &= \tfrac{1}{\sqrt{2}}\tfrac{\mu}{\lambda} + \eta \\ \phi_2 &= \tfrac{1}{\sqrt{2}}\tfrac{\mu}{\lambda} + \beta \\ A_\mu &= 0 + A_\mu \end{aligned}\right\} \text{Into } \mathcal{L}(\phi,\phi^*,A_\mu) = \frac{1}{2}\left[\left(\partial_\mu - \frac{iq}{\hbar c}A_\mu\right)\phi^*\right]\left[\left(\partial + \frac{iq}{\hbar c}\right)\phi\right] - \frac{1}{2}\mu^2\phi^*\phi + \frac{1}{4}\lambda^2(\phi^*\phi)^2 + \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$$

Note: $\phi^*\phi = \left(\frac{1}{\sqrt2}\frac{\mu}{\lambda}+\eta\right)^2 + \left(\frac{1}{\sqrt2}\frac{\mu}{\lambda}+\beta\right)^2 = \frac{\mu^2}{\lambda^2} + \sqrt2\frac{\mu}{\lambda}\eta + \sqrt2\frac{\mu}{\lambda}\beta + \eta^2 + \beta^2$

$$\mathcal{L}(\eta,\beta,A_\mu) = \frac{1}{2}\left[\left(\partial_\mu - \frac{iq}{\hbar c}A_\mu\right)\left(\frac{1}{\sqrt2}\frac{\mu}{\lambda} + \eta - i\frac{1}{\sqrt2}\frac{\mu}{\lambda} - i\beta\right)\right]\left[\left(\partial^\mu + \frac{iq}{\hbar c}A^\mu\right)\left(\frac{1}{\sqrt2}\frac{\mu}{\lambda} + \eta + i\frac{1}{\sqrt2}\frac{\mu}{\lambda} + i\beta\right)\right]$$

$$- \frac{1}{2}\mu^2\left(\frac{\mu^2}{\lambda^2} + \sqrt2\frac{\mu}{\lambda}\eta + \sqrt2\frac{\mu}{\lambda}\beta + \eta^2 + \beta^2\right)$$

$$+ \frac{1}{4}\lambda^2\left(\frac{\mu^2}{\lambda^2} + \sqrt2\frac{\mu}{\lambda}\eta + \sqrt2\frac{\mu}{\lambda}\beta + \eta^2 + \beta^2\right)^2$$

$$+ \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$$

$$= \frac{1}{2}\left(\partial_\mu\eta - \frac{q}{\hbar c}\frac{\mu}{\sqrt2\lambda}A_\mu - \frac{q}{\hbar c}\beta A_\mu\right)\left(\partial^\mu\eta - \frac{q}{\hbar c}\frac{\mu}{\sqrt2\lambda}A^\mu - \frac{q}{\hbar c}\beta A^\mu\right)$$

$$+ \frac{1}{2}\left(\partial_\mu\beta + \frac{q}{\hbar c}\frac{\mu}{\sqrt2\lambda}A_\mu + \frac{q}{\hbar c}\eta A_\mu\right)\left(\partial^\mu\beta + \frac{q}{\hbar c}\frac{\mu}{\sqrt2\lambda}A^\mu + \frac{q}{\hbar c}\eta A^\mu\right)$$

$$- \frac{\mu^4}{\lambda^2} - \cancel{\frac{1}{\sqrt2}\frac{\mu^3}{\lambda}\eta} - \cancel{\frac{1}{\sqrt2}\frac{\mu^3}{\lambda}\beta} - \cancel{\frac{1}{2}\mu^2\eta^2} - \cancel{\frac{1}{2}\mu^2\beta^2}$$

$$+ \frac{1}{4}\lambda^2\left(\frac{\mu^4}{\lambda^4} + 2\frac{\mu^2}{\lambda^2}\eta^2 + 2\frac{\mu^2}{\lambda^2}\beta^2 + \eta^4 + \beta^4 + \cancel{2\sqrt2\frac{\mu^3}{\lambda^3}\eta} + \cancel{2\sqrt2\frac{\mu^3}{\lambda^3}\beta}\right.$$

$$+ \cancel{2\frac{\mu^2}{\lambda^2}\eta^2} + \cancel{2\frac{\mu^2}{\lambda^2}\beta^2} + 4\frac{\mu^2}{\lambda^2}\eta\beta + 2\sqrt2\frac{\mu}{\lambda}\eta^3 + 2\sqrt2\frac{\mu}{\lambda}\eta\beta^2$$

$$\left. + 2\sqrt2\frac{\mu}{\lambda}\beta\eta^2 + 2\sqrt2\frac{\mu}{\lambda}\beta^3\right) + \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$$

$$= \underline{\frac{1}{2}\partial_\mu\eta\partial^\mu\eta} + \underline{\frac{1}{4}\frac{q^2}{\hbar^2c^2}\frac{\mu^2}{\lambda^2}A_\mu A^\mu} + \frac{q^2}{\hbar^2c^2}\beta^2A_\mu A^\mu - \underline{\frac{q}{\hbar c}\frac{\mu}{\sqrt2\lambda}\partial_\mu(\eta)A^\mu} - \frac{q}{\hbar c}\partial_\mu(\eta)\beta A^\mu + \frac{1}{2}\frac{q^2}{\hbar^2c^2}\frac{\mu^2}{\lambda^2}\beta A_\mu A^\mu$$

$$+ \underline{\frac{1}{2}\partial_\mu\beta\partial^\mu\beta} + \underline{\frac{1}{4}\frac{q^2}{\hbar^2c^2}\frac{\mu^2}{\lambda^2}A_\mu A^\mu} + \frac{q^2}{\hbar^2c^2}\eta^2A_\mu A^\mu + \underline{\frac{q}{\hbar c}\frac{\mu}{\sqrt2\lambda}\partial_\mu(\beta)A^\mu} + \frac{q}{\hbar c}\partial_\mu(\beta)\eta A^\mu + \frac{1}{2}\frac{q^2}{\hbar^2c^2}\frac{\mu^2}{\lambda^2}\eta A_\mu A^\mu$$

$$- \frac{3}{4}\frac{\mu^4}{\lambda^2} + \underline{\frac{1}{2}\mu^2\beta^2} + \underline{\frac{1}{2}\mu^2\eta^2} + \frac{1}{4}\lambda^2\eta^4 + \frac{1}{4}\lambda^2\beta^4 + \underline{\mu^2\eta\beta} + \frac{1}{\sqrt2}\mu\lambda\left(\eta^3 + \eta\beta^2 + \beta\eta^2 + \beta^3\right)$$

$$+ \underline{\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}}$$

The underlined terms are important. The others are either a constant or standard interaction terms.

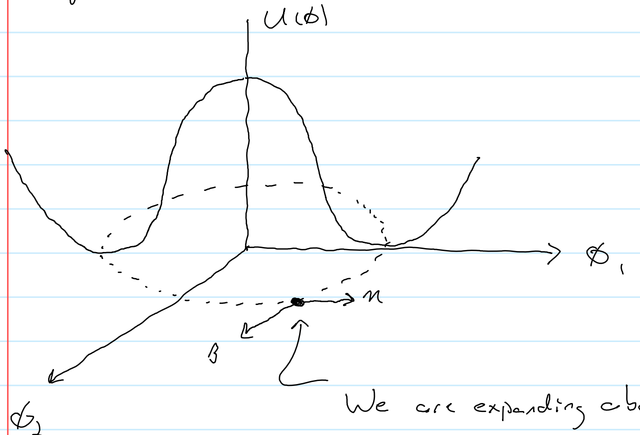
Then: $$\mathcal{L}(\eta,\beta,A_\mu) = \underbrace{\frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\mu^2\eta^2}_{\text{massive K-G}} + \underbrace{\frac{1}{2}\partial_\mu\beta\partial^\mu\beta + \frac{1}{2}\mu^2\beta^2}_{\text{massive K-G}} + \underbrace{\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(\frac{q\mu}{\hbar c\lambda}\right)^2 A_\mu A^\mu}_{\text{massive Proca}}$$

$$+ \underbrace{\frac{q}{\hbar c}\frac{\mu}{\sqrt2\lambda}\partial_\mu(\beta)A^\mu}_{\text{interconverts } \beta\leftrightarrow A^\mu} - \underbrace{\frac{q}{\hbar c}\frac{\mu}{\sqrt2\lambda}\partial_\mu(\eta)A^\mu}_{\text{interconverts } \eta\leftrightarrow A^\mu} + \underbrace{\mu^2\eta\beta}_{\text{interconverts } \eta\leftrightarrow\beta} + \text{interactions}$$

Any time we see a quadratic term w/ 2 different fields (like the 3 examples above), this implies an interaction which takes one excitation, e.g. A^μ, and spontaneously changes it to another, e.g. B. This may at first seem like a decay of $A^\mu \to B$, but for true decays there are always at least 3 states involved, e.g. $A \to B + C + \cdots$

What these quadratic interactions are actually telling is that B and A^μ are not completely distinct, nor are π and A^μ or π and B.

To get the actual fundamental fields let's consider a picture of what is going on.



We are expanding about this solution $\phi_1 = \frac{1}{\sqrt{2}}\frac{\mu}{\lambda} = \phi_2$

But our fluctuations $\phi_1 + \eta$ and $\phi_2 + \beta$ are along the ϕ_1, ϕ_2 axes.

But from lecture we know the true massive Higgs boson should be a radial fluctuation, while the massless Goldstone mode should be along the circular valley.

To see this consider $\eta' = \frac{1}{\sqrt{2}}(\eta+\beta)$, $\beta' = \frac{1}{\sqrt{2}}(\eta-\beta)$ $\Rightarrow$ $\eta = \frac{1}{\sqrt{2}}(\eta'+\beta')$, $\beta = \frac{1}{\sqrt{2}}(\eta'-\beta')$

Now let's rewrite the important part of our previous result in terms of η' and β':

$$\mathcal{L}(\eta,\beta,A_\mu) = \frac{1}{2}\partial_\mu\eta\partial^\mu\eta + \frac{1}{2}\mu^2\eta^2 + \frac{1}{2}\partial_\mu\beta\partial^\mu\beta + \frac{1}{2}\mu^2\beta^2 + \underbrace{\frac{q}{\hbar c}\frac{\mu}{\sqrt{2}\lambda}\partial_\mu(\beta)A^\mu - \frac{q}{\hbar c}\frac{\mu}{\sqrt{2}\lambda}\partial_\mu(\eta)A^\mu} + \mu^2\eta\beta$$

Note this is $-\frac{q}{\hbar c}\frac{\mu}{\sqrt{2}\lambda}\partial_\mu(\eta-\beta)A^\mu$

$$\Downarrow$$

$$\mathcal{L}(\eta',\beta',A_\mu) = \frac{1}{4}\partial_\mu(\eta'+\beta')\partial^\mu(\eta'+\beta') + \frac{1}{4}\mu^2(\eta'+\beta')^2 + \frac{1}{4}\partial_\mu(\eta'-\beta')\partial^\mu(\eta'-\beta') + \frac{1}{4}\mu^2(\eta'-\beta')^2$$

$$- \frac{q}{\hbar c}\frac{\mu}{\lambda}\partial_\mu\beta' A^\mu + \frac{1}{2}\mu^2(\eta'+\beta')(\eta'-\beta')$$

$$= \frac{1}{4}\partial_\mu\eta'\partial^\mu\eta' + \frac{1}{4}\partial_\mu\beta'\partial_\mu\beta' + \cancel{\frac{1}{2}\partial_\mu\eta'\partial^\mu\beta'} + \frac{1}{4}\mu^2\eta'^2 + \cancel{\frac{1}{4}\mu^2\beta'^2} + \cancel{\frac{1}{2}\mu^2\eta'\beta'}$$

$$\frac{1}{4}\partial_\mu\eta'\partial^\mu\eta' + \frac{1}{4}\partial_\mu\beta'\partial_\mu\beta' - \cancel{\frac{1}{2}\partial_\mu\eta'\partial^\mu\beta'} + \frac{1}{4}\mu^2\eta'^2 + \cancel{\frac{1}{4}\mu^2\beta'^2} - \cancel{\frac{1}{2}\mu^2\eta'\beta'}$$

$$- \frac{q}{\hbar c}\frac{\mu}{\lambda}\partial_\mu\beta' A^\mu + \frac{1}{2}\mu^2\eta'^2 - \cancel{\frac{1}{2}\mu^2\beta'^2}$$

$$= \underbrace{\underline{\frac{1}{2}\partial_\mu\eta'\partial^\mu\eta' + \mu^2\eta'^2}}_{\text{One massive Higgs}} + \underbrace{\underline{\frac{1}{2}\partial_\mu\beta'\partial^\mu\beta'}}_{\text{One massless Goldstone}} - \frac{q}{\hbar c}\frac{\mu}{\lambda}\partial_\mu\beta' A^\mu$$

We still have the quadratic term $\partial_\mu\beta' A^\mu$, but we know that we can use the original symmetry to "gauge away" fluctuations in β (set them to zero). This is where A^μ eats Goldstone.

BTW: Another route to getting the nasty Lagrangian for α and β would be to start with the Lagrangian given in class for α and β about $\phi_1 = \frac{\mu}{\lambda}$, $\phi_2 = 0$ and call those fluctuations α', β' then just substitute $\alpha' = \frac{1}{\sqrt{2}}(\alpha+\beta)$, $\beta' = \frac{1}{\sqrt{2}}(\alpha-\beta)$ to get the Lagrangian on the previous page :)

2. For $\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 + \frac{1}{2}\partial_\mu\phi_3\partial^\mu\phi_3 - \frac{\mu^2}{2}(\phi_1^2+\phi_2^2+\phi_3^2) + \frac{\lambda^2}{4}(\phi_1^2+\phi_2^2+\phi_3^2)^2$

we notice that the symmetry group is transformations which leave $\phi_1^2+\phi_2^2+\phi_3^2 = \text{constant}$. But these are just rotations in the 3D space of ϕ_1, ϕ_2, ϕ_3. For fixed this means we are moving on the surface of a two-sphere, which means we should expect two Goldstone modes (compare with the circular symmetry from the $\phi^*\phi$ example in class where we got one Goldstone mode).

Of course, we still have only one radial (or Higgs) mode. To get its mass we can just choose $\phi_1 = \frac{\mu}{\lambda}$, $\phi_2 = \phi_3 = 0$ and then let $\phi_1 + \eta$ and search for quadratic terms in η.

From $-\frac{\mu^2}{2}(\phi_1^2+\phi_2^2+\phi_3^2)$ we expect $-\frac{\mu^2}{2}\eta^2$

From $\frac{\lambda^2}{4}(\phi_1^2+\phi_2^2+\phi_3^2)^2$ we expect $\frac{\lambda^2}{4}\left(\left(\frac{\mu}{\lambda}+\eta\right)^2+\cdots\right)^2 = \frac{\lambda^2}{4}\left(\frac{\mu^2}{\lambda^2} + \overbrace{\eta^2}^{2\times} + \overset{(\ )^2}{2\frac{\mu}{\lambda}\eta} + \cdots\right)^2$

$$= \frac{\lambda^2}{4}\left(2\frac{\mu^2}{\lambda^2}\eta^2 + 4\frac{\mu^2}{\lambda^2}\eta^2\right.$$

$$= \frac{3}{2}\mu^2\eta^2$$

So in total the Higgs fluctuation has mass term $\left(\frac{3}{2}\mu^2 - \frac{1}{2}\mu^2\right)\eta^2 = \mu^2\eta^2 = \frac{1}{2}\left(\frac{m_H c}{\hbar}\right)^2\eta^2$

$$\Rightarrow m_H = \sqrt{2}\,\frac{\hbar\mu}{c}$$