The weak interactions have many peculiar features that set them apart from QCD and ETH:

1. Every single matter particle in the SM exhibits weak interactions (both charged for ETH, only quarks for QCD)
2. The force mediators are massive (unlike photons and gluons)
3. The weak interactions violate parity, charge conjugation and CP.
4. The weak interactions can change “flavor”, i.e. particle type ⇒ responsible for decay!

Perhaps the strongest part of the weak interactions is that they are not realized as a symmetry of the SM, at least not at room temperature type energies. We will eventually explain what this means and in fact this will solve the neutrino mass issue by bringing in the Higgs mechanism. But more on that later.

To keep in line with our development of ETH and QCD as theories of local (or gauge) invariance, we will go ahead and formulate the weak interactions in terms of a gauge symmetry. This is relevant since at some point in the history of the universe this is how it appeared. More on that later!

Perhaps the most surprising feature of formulating the weak interactions in terms of a gauge symmetry is that to do so, we are forced to ”unify” the weak force with ETH!
Electroweak Unification

It is often said that the total gauge symmetry group of the SM is $SU(3) \times SU(2) \times U(1)$.

However, this is not quite right. The correct groups are:

- $SU(3) \times SU(2) \times U(1)$ (High Energy)
- $SU(2) \times U(1)$ (Low Energy)

So we need to start with $SU(2) \times U(1)$. The "L" in $SU(2)$ means that this symmetry is only relevant for "left-handed" fermion states. This is a bit of a misnomer since handedness has to do with helicity, whereas in actuality the $SU(2)$ acts on states of definite chirality (which does match helicity for massless particles).
Recall: $\mathfrak{L}_{\text{bare}} = (k_c) \sum [Y^\dagger \gamma_\mu \partial_\mu + \kappa \frac{1}{2} Y^\dagger Y$

\[ = k_c (\frac{1}{2} \gamma_\mu \partial_\mu + \gamma_\mu g^a \partial_\mu \lambda^a + \gamma_\mu g^a \partial_\mu \lambda^a) \]

\[ = \gamma^\mu (\sigma_1 \partial_\mu \partial_\mu - \sigma_2 \partial_\mu \partial_\mu) \]

\[ = \gamma^\mu (I \pm \sigma_2) \]

\[ \theta^a = (I \pm \sigma_2) \]

\[ \psi^a = \psi^a \]

\[ P_\pm = \frac{1}{2} (\pm \gamma^\mu \partial_\mu) \psi^a \]

\[ P_\pm = \frac{1}{2} (\pm \gamma^\mu \partial_\mu) \psi^a \]

Note that: $\psi^a \gamma^\mu \psi^a = \frac{1}{2} \gamma^\mu \psi^a \gamma^\mu \psi^a$

\[ \Rightarrow \text{All derivative terms mixing } L \text{ and } R \text{ vanish.} \]

\[ \overline{\psi}_R \gamma^\mu \psi_L = \overline{\psi}_R \gamma^\mu \psi_L \]

\[ \Rightarrow \text{All mass terms mixing } L \text{ and } R \text{ vanish.} \]

\[ \mathfrak{L}_{\text{bare}} = (k_c) \sum [Y^\dagger \gamma_\mu \partial_\mu + \kappa \frac{1}{2} Y^\dagger Y + \kappa \frac{1}{2} Y^\dagger Y + \kappa \frac{1}{2} Y^\dagger Y] \]
L_{\text{lep}} = (k_\text{e}) \bar{e}_R Y^\nu e_L + (k_\text{\bar{e}}) \bar{\nu}_R \bar{\nu}_L + \nu^- \bar{\nu}_R + \nu^+ \bar{\nu}_L

Now that we have \( L + R \) in the gauge, we need a "doublet" for the \( SU(2)_L \) to act on.

Recall for quarks we introduced a triplet \( t = \left( \begin{array}{c} t^e \\ t^\nu \\ t^\tau \end{array} \right) \) for \( SU(3) \).

We actually pair particles into left-handed doublets: \( \chi_L = \left( \begin{array}{c} \nu_e \\ \nu_\mu \\ \nu_\tau \\ \nu_L \\ \tau_L \end{array} \right) \), \( \chi_R = \left( \begin{array}{c} \nu_e \\ \nu_\mu \\ \nu_\tau \\ \nu_L \\ \tau_L \end{array} \right) \).

And have right-handed singlets: \( \nu_\tau, \nu_\mu, \nu_e, \nu_\tau, \nu_\mu, \nu_e, \nu_\tau, \nu_\mu, \nu_e \).

What about \( \nu_R, \nu_R, \nu_R \)? They don't exist! At least not for the massive neutrino story.

For simplicity, we will focus on the first generation of leptons, i.e. \( \chi_L = \left( \begin{array}{c} \nu_e \\ \nu_\mu \\ \nu_\tau \end{array} \right) \), \( \chi_R \) and ignore the mass term since it never plays a role in generating interactions.

\[ L_{\text{lep}} = (k_\text{e}) \bar{\chi}_L Y^\nu \chi_L + (k_\text{\bar{e}}) \bar{\nu}_R \bar{\nu}_L \]
\[ L_{\text{WIMP}} = (k_c) \bar{\chi}_L \gamma^\mu D_\mu \chi_L + (k_c) \bar{\chi}_L \gamma^\mu \gamma_5 \chi_L \]

This is invariant under global \( SU(6) \times SU(3)_W \) where \( SU(6) \) is

\[
\text{intermediate} \quad \psi \xrightarrow{c} \chi_L, \quad \bar{\psi} \xrightarrow{\bar{c}} \bar{\chi}_L
\]

\( SU(3)_W \) is

\[
\text{weak hypercharge} \quad U(1)_Y
\]

Each of these can be promoted to a local symmetry of the Lagrangian by the methods we have covered in the preceding lectures.

Let:

\[
\partial_\mu \chi_L \rightarrow D_\mu \chi_L = \partial_\mu \chi_L + ig^2 \frac{\sigma^{\mu\nu}}{2} \chi_L \partial_{\nu} \phi
\]

\( g, g_\phi \) are new fields

\[
\partial_\mu \phi \rightarrow D_\mu \phi = \partial_\mu \phi + ig^2 \frac{\sigma^{\mu\nu}}{2} \chi_L \partial_{\nu} \phi
\]

\( U_\mu \phi = e^{-ig \frac{2m}{g^2} \partial_\mu \phi} + \frac{i}{g^2} \partial_\mu (e^{ig \frac{2m}{g^2} \partial_\phi}) e^{-ig \frac{2m}{g^2} \partial_\phi}
\]

Gauge field transformation rules

\[
B_\mu = B_\mu + \partial_\mu \phi
\]

Thus for each gauge field we introduce a kinetic term using \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \)

and again this would give the usual \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) for the \( U(1)_Y \)

\( F_{\mu\nu} = \partial_\mu U_{\nu} - g_\phi \bar{\chi}_\nu U_{\mu} - g U_{\mu} \bar{\chi}_\nu \) for \( SU(3)_W \)

The latter will include interactions between the \( SU(3)_W \) and \( SU(6) \) gauge bosons (just like the group of QCD).

The \( SU(6) \) interactions have a particularly interesting structure. We expect 3 gauge bosons \( \omega^+, \omega^0, \omega^- \) which act on the left-handed doublets, e.g. \( \mathbf{X}_L = (\mathtt{e}_L, \nu_L) \).

Using the machinery of good old spin-1 from QM (that involved \( SU(2) \) as well!) we can think of this in terms of:

\[
\omega^+ = \frac{\sigma^{\mu\nu}}{2} \rightarrow \omega^+ \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = \omega^+ \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad \omega^+ \left( \begin{array}{c} 0 \\ 1 \end{array} \right) = \frac{1}{2} \left( \begin{array}{c} 1 \\ 1 \end{array} \right)
\]

\[
\omega^- = \frac{\sigma^{\mu\nu}}{2} \rightarrow \omega^- \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = \omega^- \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad \omega^- \left( \begin{array}{c} 0 \\ 1 \end{array} \right) = \frac{1}{2} \left( \begin{array}{c} 1 \\ 1 \end{array} \right)
\]

\[
\omega^0 = \sigma^{\mu\nu} \partial_\nu \phi = \omega^0 \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = \omega^0 \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad \omega^0 \left( \begin{array}{c} 0 \\ 1 \end{array} \right) = \frac{1}{2} \left( \begin{array}{c} 1 \\ 1 \end{array} \right)
\]

But notice that \( \omega^+ (\omega^-) \) changes \( \mathtt{e} \rightarrow \nu L \) (\( \nu L \rightarrow e \)) respectively, so clearly the \( \pm \) here must be connected to electric charge. Their gauge bosons mediate interactions which change the electric charge of the matter involved.

The \( \omega^0 \) on the other hand does not change the particle type, hence does not change the electric charge, i.e., it is neutral.
There are 2 key problems:

1) We know that the weak gauge bosons are massive, and we already know that the Peskin mass term \( (\Xi^l)^* \Xi^l \) is not gauge invariant.

2) Recall that to have mass terms for spins two, both the L and R parts of \( \Phi \) had to combine, e.g. \( \Phi L \Phi R \).
    However, we have just constructed a gauge theory where the L and R parts transform differently.
    There is no way we can expect \( \Phi L \Phi R \) to be gauge invariant!

Both of these problems will be solved with the Higgs mechanism for mass generation. An crucial part of this process
is the breaking of \( SU(2)_L \times U(1)_y \rightarrow U(1)_{EM} \).

We will have much more to say about how such a symmetry can be broken, but for now we will just highlight
the implication for the electromagnetic interactions.

\( SU(2)_L \times U(1)_y \) has 4 generators \( \pm \gamma_3, \pm \beta \), \( \tilde{B}_\mu \). After symmetry breaking to \( U(1)_{EM} \) we only expect one symmetry generator to survive. Which one is it?

You might have thought it would be \( B_\mu \), then certainly \( U(1)_{EM} \), but that is not the case.

In actuality, this \( B_\mu \) "mixes" with the neutral \( \gamma \) from \( SU(2)_L \). We can form 2 orthogonal states:

\[
\begin{align*}
A_\mu &= \bar{B}_\mu \cos \theta_W + \gamma_3 \sin \theta_W \\
Z_\mu &= \bar{B}_\mu \sin \theta_W + \gamma_3 \cos \theta_W
\end{align*}
\]

\( \theta_W \) = Weinberg mixing angle

So it should be clear that we cannot identify just \( U(1)_{EM} \) with the \( U(1) \) factor in \( SU(2)_L \times U(1)_y \)!