If at some point we had summed over $\omega \Rightarrow \varphi - nc$.

Note: There are no spinors left in the expression! Only $\varphi_0$'s and $\psi$'s!

To impose sum over $S_0, S_x$ replace
\[
\bar{u}(a) \Gamma_{\mu(a)}^\dagger \Gamma_{\nu(a)}^\dagger \Gamma_{\mu(b)} \Gamma_{\nu(b)}^\dagger = \text{Tr} \left[ \Gamma_{\mu(a) + \nu(a)} \Gamma_{\mu(b) + \nu(b)} \right]
\]

Let's put this result to work:

\[
e^+ + n \rightarrow e^+ + \nu
\]

\[
H = \frac{-g^2}{(\lambda - \lambda_2)} \bar{u}(3)^\dagger \gamma^\mu u(1)^\dagger \bar{u}(1)^\dagger \gamma^\nu u(2)^\dagger q_{\mu\nu}
\]

\[
\langle H^I \rangle = \frac{g^2}{(\lambda - \lambda_2)} \text{Tr} \left[ \gamma^\mu (\beta + m \gamma^5) \gamma^\nu (\beta + m \gamma^5) \right] q_{\mu\nu}
\]

\[
\langle H^I \rangle = \frac{g^2}{(\lambda - \lambda_2)} \text{Tr} \left[ \gamma^\mu (\beta + m \gamma^5) \gamma^\nu (\beta + m \gamma^5) \right] q_{\mu\nu}
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\]

\[
\langle H^I \rangle = \frac{g^2}{(\lambda - \lambda_2)} \text{Tr} \left[ \gamma^\mu (\beta + m \gamma^5) \gamma^\nu (\beta + m \gamma^5) \right] q_{\mu\nu}
\]
To continue, we need to know how to evaluate traces in spin space. Fortunately, there are some useful results:

1. \( Tr(\gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\lambda) = 4(\gamma^\rho g_{\lambda \sigma} - g_{\gamma \sigma} \gamma^\rho + g_{\gamma \lambda} \gamma^\rho) \)

2. \( Tr(\gamma^\nu \gamma^\rho \gamma^\lambda) = 0 \)

3. \( Tr(\gamma^\nu \gamma^\lambda) = 4 \gamma^\nu \gamma^\lambda \gamma^\rho = \gamma^\nu \) (e.g., \( \mu \nu \rho \lambda \) is totally symmetric)

Then:

\[ Tr[\gamma^\nu (\gamma^\rho + \gamma^\sigma) (\gamma^\lambda + \gamma^\mu)] = 0 \]

Using:

\[ Tr[\gamma^\nu \gamma^\rho] = 4 \gamma^\nu \gamma^\rho = 4 \gamma^\nu \gamma^\rho \]

Thus, in space where \( M \) is a matrix:

\[ X = \frac{1}{\sqrt{16}} \left[ \gamma^\nu \gamma^\rho \gamma^\lambda \gamma^\mu \right] = \frac{1}{\sqrt{16}} \left[ \gamma^\nu \gamma^\rho \gamma^\lambda \gamma^\mu \right] = \frac{1}{\sqrt{16}} \left[ \gamma^\nu \gamma^\rho \gamma^\lambda \gamma^\mu \right] \]

Thus, for our commutator result:

\[ \langle [\gamma^\rho, \gamma^\nu] \rangle = \frac{1}{\sqrt{16}} \left[ \gamma^\nu \gamma^\rho \gamma^\lambda \gamma^\mu \right] \]

Note: The first expression is in terms of only \( \gamma^\nu \gamma^\rho \), i.e., \( \gamma^\mu \gamma^\lambda \)!
Recall that for 2-body scattering: 
\[ \frac{d\sigma}{d\Omega} = \left( \frac{\alpha}{4\pi} \right)^2 \frac{5}{3} \left( \frac{m}{E_1} \right)^2 \frac{1}{\hat{p}_1 \cdot \hat{n}} \] 
for the CM-frame.

If we then take the approximation \( \hbar \gg m \) and assume that \( E_1 = E_2 \ll \hbar c^2 \), we find:
\[ \hat{p}_1 = \hat{p}_2 \; \text{and} \; E_1 + E_2 = \hbar c^2 \]

The CM-frame is essentially the rest frame of \( \gamma \).

Then:
\[ \langle \frac{d\sigma}{d\Omega} \rangle = \left( \frac{\alpha}{4\pi} \right)^2 \langle \gamma \rangle \]

\[ E_1, \hat{p}_1 \rightarrow \gamma \]

\[ \Rightarrow \quad \hat{p}_1 \cdot E \]

\[ \hat{p}_1 = (\hbar c, 0) \quad \hat{p}_2 = \left( \frac{E_1}{c}, \hat{p}_1 \right) \quad \hat{p}_3 = (\hbar c, \hat{p}_1) \quad \hat{p}_4 = \left( \frac{E_1}{c}, \hat{p}_1 \right) \]

\[ \text{I approximate since } \hbar c \gg m \]

\[ \langle \gamma \rangle = (\hat{\gamma}, \hat{p}_1 \cdot \hat{p}_1)^2 \approx 0 - (\hat{\gamma}, \hat{p}_1 \cdot \hat{p}_1) \approx -\hat{\gamma} - \hat{p}_1 + \hat{\gamma} \hat{p}_1 \]

\[ = -\frac{\hbar}{c} (1 - \cos \theta) \]

\[ = -\frac{\hbar}{c} \frac{1}{2} \sin^2 \frac{\theta}{2} \]

\[ \hat{p}_1 \cdot \gamma \approx \left( \frac{E_1}{c} \right)^2 - \hat{p}_1 \cdot \gamma \approx \hbar c + \hat{p}_1 \cdot \gamma \approx \hbar c \]

\[ \langle \frac{d\sigma}{d\Omega} \rangle = \left( \frac{\alpha}{4\pi} \right)^2 \frac{5}{3} \left( \frac{m}{E_1} \right)^2 \frac{1}{\hat{p}_1 \cdot \hat{n}} \]

\[ \text{Note: } q_e = \pm \frac{e}{\hbar c} = \pm \frac{e}{\hbar c} \]

In the non-relativistic limit \( \frac{\hbar}{c} \ll (\hbar c)^2 \) this becomes:
\[ \frac{d\sigma}{d\Omega} = \left( \frac{e^2}{8\pi \hbar c} \right)^2 \left( \frac{m}{E_1} \right)^2 \frac{1}{\hat{p}_1 \cdot \hat{n}} \]

\[ \text{Rutherford Formula} \]
The effects of virtual particle pairs start at 4th order with the largest contribution being:

\[ \frac{g^4}{q^4} \left[ \frac{\text{Tr} \left[ \bar{\psi}(k+\not{q})\gamma^\mu \psi(k+\not{q}) \right]}{(k^2-m^2)^2} \right] \frac{1}{(k-q)^2} \]

This contribution is actually divergent and will eventually lead us to the topic of renormalization.

We also get another “rule of thumb” associated with the new Feynman rule. 
Furry’s Theorem: When constructing diagrams you can ignore contributions from closed internal matter loops w/ an odd # of vertices (since \( \not{v} = 0 \)).

An application is photon “decay”:

\[ \rightarrow \not{A} = 0 \]