Renormalization

We know that after using all 5-functions: \( M \rightarrow \Phi^5(\text{renorm.-param.}) \left\{ f(P, \mu) \right\} \frac{\delta \Phi}{\delta \mu} \text{renorm.} \)

In many scenarios the remaining \( \Phi \) diverges. This is usually at large \( q_k \).

We can render the \( \Phi \) finite by "regularizing" i.e., e.g., introduce factors \( \frac{f}{q_k^2 - A^2} \) \( \text{Pauli-Villars massive regulator} \)

where \( \Phi \) is a free parameter (set related to \( M_k \)).

In the end we take \( \Phi \rightarrow 0 \) so factor \( \rightarrow 1 \). If we're lucky then the resulting form of the integral will split into two parts: One will depend on \( q_k \) and \( \rightarrow \infty \) (as \( A \rightarrow 0 \)).

One will be ind. of \( q_k \) and remain finite.

If we're really lucky the \( \Phi \)-dep. parts will look like:

\[ M_\text{p} = M_\text{q} + \Phi^\text{reg.} (\Phi) \]

\[ q_\text{p} = q_\text{q} + q_\text{reg.} (\Phi) \]

In this case we can say:

\( M_\text{p} \) and \( q_\text{p} \) are the inputs in our original theory. We naïvely expected these to be the finite values measured in experiments. That was dumb!

\( M_\text{q} \) and \( q_\text{q} \) are the true "bare" values that should fundamentally define the theory, but what we measure are the "physical" values \( M_\text{p} \) and \( q_\text{p} \).

We knew \( M_\text{q} \) and \( q_\text{q} \) are finite, but \( M_\text{q} \) and \( q_\text{q} \) could be anything, in particular divergent enough to cancel \( M_\text{p} \) and \( q_\text{p} \)!

If this story plays out, then we call the theory renormalizable, \( M \rightarrow \text{Ren} \). \( \Phi \rightarrow \text{Non-Ren}. \)

Note: You can always regularize the theory to make it finite. The key to renormalizability is removing the regulator in a consistent way.

Note to note: Consistent QED theories have built-in regulators as part of the theory (no need to remove).
Renormalization

We want to explore the "divergence" of real electric charge by virtual particle pairs in a more quantitative way. This will lead us to understand how to handle many divergences encountered in Feynman amplitudes.

Consider $e + e^\prime ightarrow e + e^\prime$ (so that switching outgoing particle logs).

At 2nd order we have the "tree-level" contribution (simplest nontrivial):

$$\Gamma_{\text{tree}} = -\frac{\alpha}{q^2} \left[ \overline{\psi}(x)\gamma^0 \psi(x) \right] \frac{g\nu}{q^2} \left[ \overline{\psi}(x')\gamma^\nu \psi(x') \right]$$

The effects of virtual particle pairs start at 4th order with the largest contribution being:

$$\Gamma_{\text{imp}} = -\frac{\alpha}{a^2} \left[ \overline{\psi}(x)\gamma^0 \psi(x) \right] \left[ \overline{\psi}(x')\gamma^\nu \psi(x') \right] \frac{g\nu}{(k-q)^2} \left[ \overline{\psi}(x')\gamma^\nu \psi(x') \right]$$

Call this $\Gamma_{\text{imp}} = \Gamma(q)$

After some massaging:

$$\Gamma(q) = -\frac{\alpha}{a^2} \left\{ \frac{d}{dz} - \frac{1}{2} \right\} \left[ \overline{\psi}(x)\gamma^0 \psi(x) \right] \left[ \overline{\psi}(x')\gamma^\nu \psi(x') \right] \left\{ \overline{\psi}(x')\gamma^\nu \psi(x') \right\}$$

The next step in the "renormalization" program is to "regularize" the divergence (make it a finite contribution). We will simply use an upper "cut-off":

$$\int_{\lambda^2}^{M^2} \frac{dZ}{Z} = \ln \left( \frac{M^2}{\lambda^2} \right)$$

Then:

$$\Gamma(q) = -\frac{\alpha}{a^2} \left\{ \ln \left( \frac{M^2}{\lambda^2} \right) - \frac{a^2}{2} \right\}$$

Note that divergence is still present when $\lambda \rightarrow 0$.

And:

$$\Gamma_{\text{net}} = \Gamma_{\text{tree}} + \Gamma_{\text{imp}}$$

We have:

$$\Gamma_{\text{net}} = -\frac{\alpha}{a^2} \left[ \overline{\psi}(x)\gamma^0 \psi(x) \right] \frac{g\nu}{a^2} \left[ \overline{\psi}(x')\gamma^\nu \psi(x') \right]$$

where $\overline{\psi}(x)\gamma^0 \psi(x)$
Now we ask the most important question: “What is $g_e$?” We know $g_e = e \sqrt{\frac{e}{4\pi}}$

but what value do we use?

If we use the value of $e$ measured in experiments, then the result implies that $g$ diverges when $k\approx 1$ (some cutoff). However, what should go in for $e$ is the “fundamental” or “bare” value of $e$. But we don’t have access to this!

We measure:

\[ \text{screening} \]

But interpret our measurement “classically”, i.e.

\[ \text{no screening} \]

so we really measure an “effective” charge that includes all loop contributions.

To formulate this we simply rewrite our result in terms of the effective or “renormalized” charge/coupling $g_R$:

\[ g_R(q^2) \equiv g_e \sqrt{1 - \frac{k^2}{4\pi}} \left[ \ln \left( \frac{k^2}{\Lambda^2} \right) - \frac{1}{8\pi} \right] = \text{physically measured value of } e \sqrt{\frac{e}{4\pi}} \text{ at experiment} \]

This is the “bare” value of $e$.

Then of course: \[ \text{Higgs mass} = -g_R(q^2) \left[ \alpha_c(q^2) \right] \left[ \alpha_s(q^2) \right] \]

which looks like it came from $\alpha_s$ alone!

This is an example of an “effective” theory where the quantum loop corrections are bundled into renormalized quantities and the result is interpreted as the “classical” tree-level, i.e.

\[ \gamma = \gamma_{\text{class}} g_R(q^2) \]

This is often written in terms of the zero-momentum (large distance) value:

\[ g_R(0) = g_e \sqrt{1 - \frac{k^2}{4\pi}} \left[ \ln \left( \frac{k^2}{\Lambda^2} \right) \right] \]

to obtain:

\[ g_e(q^2) = g_R(0) \sqrt{1 + g_R(0) \frac{\alpha_s(0)}{12\pi} f\left( \frac{q^2}{\Lambda^2} \right)} \]

Note: \[ g_R(0) = g_e (1 - \frac{\alpha_s(0)}{4\pi}) = g_e - \theta(q^2) + O(q^2) \]

So at 4th order $g_R(0) \approx g_e$, which is why we use $g_R(0)$, thank!

Three important points:

1. In principle this should be done to all orders (number of loops) to get the correct $g_R(q^2)$, but lower orders dominate.
2. We encountered $\alpha_0$, but now realize that this was due to incorrectly using the measured electron charge where we should be using the unknown, bare value. This situation comes up in many QFTs, and when we can “fix” it like this, we call the theory renormalizable. The standard model is, but perturbative quantum gravity is not.
3. Even when the problem of $\alpha_0$ is solved the need for renormalizing, because the truth is that measured values of “constant” actually depend on $q$, i.e. they run (running couplings).

Often people will leave out the $q^2$ dependence and work in terms of $g_R(0)$. But then they must include all loop corrections. But there will now be finite since they are in terms of $g_R(0)$ and not $g_e$.

Finally, as a check:

\[ g_R(q^2) = g_R(0) \sqrt{1 + \frac{\alpha_s(0)}{12\pi} f\left( \frac{q^2}{\Lambda^2} \right)} \]

This increase as $q^2$ increases with $\alpha_s$, which is what we expect for a “screened” fundamental charge.
We considered \( e^+ e^- \) but there are other 4th order (1-loop) diagrams. Each play certain roles as described:

- Electron self-energy corrections
- Vertex correction
- Renormalizes electron mass
- Renormalizes electron magnetic moment

These 3 separately contribute to the electron charge renormalization (by \( \frac{\alpha}{\pi} \)) but the combined effect of these three cancels, so only \( \frac{\alpha}{\pi} \) plays a nontrivial role.

This is a good things since the corrections to the charge from these three diagrams would be mass dependent leading to different effective charges for \( e, \mu, \tau \). But we observe the same effective charge which reflects that their contributions cancel. Note: \( \frac{\alpha}{\pi} \) is mass independent. We could include \( \alpha m \), but this would renormalize the \( e, \mu, \tau \) charges the same.
In QED we found that vacuum polarization led to effective charge screening which made the electric charge "run" to larger values at smaller distances (or at larger 1/q^2 momentum transfer).

That is:
\[\alpha(1/q^2) = \alpha(0) \left[ 1 + \frac{\alpha(0)}{2\pi} \ln \left( \frac{1/q^2}{\alpha(0)} \right) \right]\]
for \(1/q^2 >> \alpha(0)\).

We can actually sum over all contributions of this form since the series ends up being geometric;
\[\alpha(1/q^1) = \frac{\alpha(0)}{1 - \frac{\alpha(0)}{2\pi} \ln \left( \frac{1/q^1}{\alpha(0)} \right)} \quad 1/q^1 >> \alpha(0)\]

Note: This does not count all possibilities, e.g., but there are the leading terms at each order.

Now \(\alpha(1/q^1)\) increases w/ increasing 1/q^2 and even blows up when \(\ln \left( \frac{1/q^2}{\alpha(0)} \right) = \frac{2\pi}{\alpha(0)} = \frac{2\pi}{\alpha(0)}\) but this is at \(10^{20}\) eV so not too troubling.

In QCD we have \(g^2\) corrected by \(g^3\) which also leads to screening,

but additionally we have \(g^4\) and \(g^5\) and \(g^6\) which leads to anti-screening.

Now which effect wins depends on the number of different quark flavors (screening) \(f\), and the number of colors \(n\) which impacts both the number of quarks (screening) and gluons (anti-screening).

Evaluating loop diagrams in QCD requires \(F^n\) gluons and \(g\) beyond us, but the result analogous to the QED case is:
\[\alpha_s(1/q^3) = \alpha_s(1/q^3) \left[ 1 + \frac{\alpha_s(1/q^3)}{2\pi} \ln \left( \frac{1/q^3}{\alpha_s(1/q^3)} \right) \right]\]
for \(1/q^3 >> \alpha_s(0)\).

This is some scale at which we set a reference value of \(\alpha_s\) (similar to \(\alpha(0)\) in QED, but we can't use \(\alpha_s(0)\) since it blows up).

Now if \(\ln (-\alpha) > 0\) anti-screening wins \(\alpha_s(1/q^3)\) decreases w/ increasing \(1/q^3\).

In the SH \(n = 3, f = 6 \Rightarrow 33-12 > 0\) so anti-screening wins and \(\alpha_s(1/q^3)\) decreases w/ increasing \(1/q^3\). This leads to asymptotic freedom which allows us to effectively use perturbation theory in QCD at short distances, e.g., inside mesons and baryons.
Of course one could try to extrapolate from this that at low $\mho$ or large distances the $\gamma(\mho)$ to and hence explain confinement, but in this regime perturbation theory breaks down.

As a side note, theorists like to play around with "pure" QCD (only gluons) since it is actually a finite theory (no renormalization needed).