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This time: Complex numbers, planes and manifolds, tensors and holonomy

Next time: Complex differential forms, Chern class, Kähler geometry and Calabi-Yaus

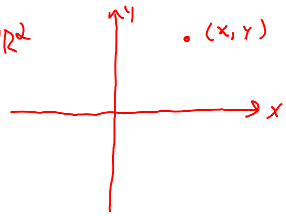
Complex Numbers

$i = \sqrt{-1}$ $\quad c = x + iy \quad x, y \in \mathbb{R}$ $\quad |c|^2 = cc^* = x^2 + y^2$

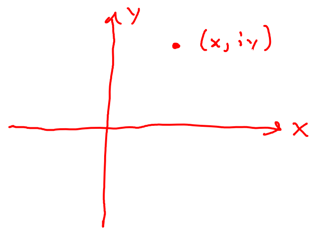
$c^* = x - iy \quad c, c^* \in \mathbb{C}$

The Complex Plane

$\mathbb{R}^2$



$\mathbb{C}$



$\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ 3D $\qquad \mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ 4D

$\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ 4D

Clearly we can exchange any $\mathbb{C}^n$ for $\mathbb{R}^{2n}$ and vice-versa.

But both of these are topologically nontrivial (∞ in all directions) and flat.

Things get more interesting when things get compact ("finite" in size) and/or curved.

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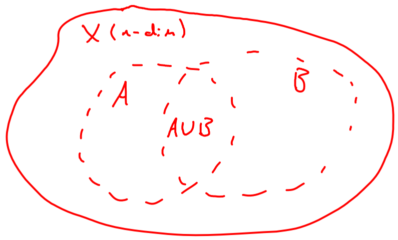
For compact or curved real spaces, we tend to focus on a special subset called "manifolds".
Roughly speaking, manifolds are smooth spaces that locally resemble $\mathbb{R}^n$, e.g. S^2.

To be precise:

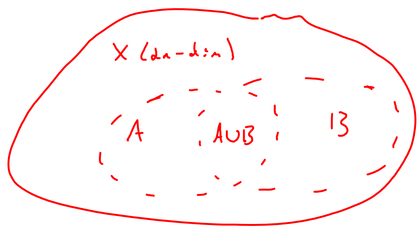


On each of A, B we have $\left.\begin{array}{l}\phi_A : A \to \mathbb{R}^n \\ \phi_B : B \to \mathbb{R}^n\end{array}\right\}$ local coordinates z^n

On $A \cup B$ we have $\phi_A \circ \phi_B^{-1} : \mathbb{R}^n \to \mathbb{R}^n \in C^p$ for a p-smooth manifold (usually $p = \infty$)

Examples: $\mathbb{R}^1 \longleftrightarrow$ a manifold
$|\mathbb{R}^1| \longrightarrow$ not a manifold

For the complex case we have:



In this case, the manifold locally looks like $\mathbb{C}^n$.

$\phi_A : A \to \mathbb{C}^n$
$\phi_B : B \to \mathbb{C}^n$
On $A \cup B$ we have $\phi_A \circ \phi_B^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ depend only on z^n (<u>not</u> z^{n*})
These are "holomorphic" maps.
(holomorphic in each z^n)

The holomorphicity of transition functions is a big part of what sets complex manifolds apart from real ones.

You may be more familiar with this in the case of $\mathbb{C}$ where $f : \mathbb{C} \to \mathbb{C}$ is a function $f(x+iy) = u(x,y) + i v(x,y)$ satisfying $\left.\begin{array}{l}\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\end{array}\right\}$ Cauchy-Riemann equations

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The point here is that $\mathbb{C}^n$ can be thought of as a space w/ $2n$ real coordinates, but with "extra", i.e. more rigid in this case, structure.

A loose analogy is: $\mathbb{R}^4$ w/ $\underbrace{(ict, x, y, z)}_{\text{complex}}$ and $\underbrace{g_{\mu\nu} = I}_{\text{trivial}}$ vs $\mathbb{M}^4$ w/ $\underbrace{(ct, x, y, z)}_{\text{real}}$ and $\underbrace{g_{\mu\nu} = \eta_{\mu\nu}}_{\text{nontrivial}}$

An interesting point is that though we can always think of a complex manifold as a real manifold w/ extra structure, we cannot take any old real manifold and just add this structure to make it complex, there are some conditions that must be met.

1. Clearly this will not work for real manifolds of odd dimension!
2. Let $J^\mu{}_\nu$ be a $(1,1)$-tensor field that satisfies $\underbrace{J^\mu{}_\nu J^\lambda{}_\mu = -\delta_\nu{}^\lambda}$. This is our higher dimensional generalization of $i^2 = -1$

   $J^\mu{}_\nu$ is called an almost complex structure and can be defined for any real even dimensional manifold.

   Now construct the $(1,2)$-tensor:

   $$N^\mu{}_{\nu\lambda} = J^\beta_\nu(\partial_\beta J^\mu{}_\lambda - \partial_\lambda J^\mu_\beta) - J^\beta_\lambda(\partial_\beta J^\mu_\nu - \partial_\nu J^\mu_\beta)$$

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   If $N^\mu_{\nu\lambda} = 0$, then $J^\mu{}_\nu$ is a complex structure and we can view the manifold in question as complex.

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General Tensors

We have seen $J^\mu{}_\nu$ which we can think of as a map from $\underbrace{T_pM}_{\substack{\text{the tangent space} \\ \text{to } M \text{ at point } p}} \to T_pM$, i.e. it acts on a vector and returns a vector.

At this point though, T_pM is a real vector space at p.
To allow complex vector-valued functions, we need a complex vector space at each p.

So consider $T_pM \otimes \mathbb{C}$.

Now the map $J^\mu{}_\nu$ acts as $J^\nu{}_\mu J^\mu{}_\alpha v^\alpha = -\delta^\nu{}_\alpha v^\alpha = -v^\alpha$ or $J^2 v = -v$, so the eigenvalues of J (on v) in $\underbrace{T_pM \otimes \mathbb{C}}_{\sim \mathbb{C}^{2n}}$ are $\pm i$.

We can call $T_p^{(1,0)}M$ and $T_p^{(0,1)}M$ the eigenspaces w/ eigenvalues $+i$ and $-i$ respectively.

Each of these eigenspaces are isomorphic to $\mathbb{C}^n$, complex conjugate to each other, and in fact we can decompose $\underbrace{T_{\mathbb{C}}M}_{\substack{\text{short for} \\ T_pM \otimes \mathbb{C}}} = \underbrace{T_p^{(1,0)}M}_{\substack{\text{holomorphic} \\ \text{tangent bundle}}} \oplus \underbrace{T_p^{(0,1)}M}_{\substack{\text{antiholomorphic} \\ \text{tangent bundle}}}$

We can similarly decompose the cotangent bundle into $T^*_{\mathbb{C}}M = T^{*(1,0)}M \oplus T^{*(0,1)}M$

Finally we can consider holomorphic vectors v^a and antiholomorphic vectors $v^{\bar{a}}$, with the decomposition of a general vector $v^\mu = v^a + v^{\bar{a}}$. We can similarly treat all vector and dual vector indices of more general tensors.