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8:05 AM

Last time we learned that we can take the complexified tangent bundle $T_{\mathbb{C}}M$ and split it into eigenspaces of J w/ eigenvalues $\pm i$. Explicitly, for $X \in T_{\mathbb{C}}M$:

$$T^{(1,0)}(M) = \{ X - iJX \} \qquad \text{e.g. } J(X - iJX) = JX - iJ^2X = JX + iX = i(X - iJX)$$
$$T^{(0,1)}(M) = \{ X + iJX \}$$

We can also easily see that $T^{(1,0)}(M) = \underbrace{\tfrac{1}{2}(1 - iJ)}_{P_-}X$ w/ $P_-^2 = \frac{1}{4}(1 - iJ - iJ + 1) = P_-$
hence P_- is a projection operator

Also recall $J^a{}_b J^b{}_c = -\delta^a{}_c$
For $\mathbb{C}^1$ we must have $a=b=c$ so this becomes $J^2 = -1$ or $i^2 = -1$.

In the end we have $T(M) = T^{(1,0)}(M) \oplus T^{(0,1)}(M)$
and the cotangent or dual version $T^*(M) = T^{*(1,0)}(M) \oplus T^{*(0,1)}(M)$.

An almost complex structure is promoted to a complex structure when $[T^{(1,0)}(M), T^{(1,0)}(M)] \subseteq T^{(1,0)}(M)$, that is when the Lie bracket of holomorphic vectors (which measures the rate of change of one vector field along the flow induced by the other) remains holomorphic. If $v = v^a\partial_a$, $w = w^b\partial_b$ then $[v,w] = v^a\partial_a w^b\partial_b - w^a\partial_a v^b\partial_b$.

For $W = \underbrace{[X - iJX, Y - iJY]}_{\subseteq T^{(1,0)}(M)} = [X,Y] - [JX,JY] - i([X,JY] + [JX,Y])$
we have that $W \subseteq T^{(1,0)}(M)$ if $JW = iW$ or
$$JW = J[X,Y] - J[JX,JY] - i(J[X,JY] + J[JX,Y])$$
we want this equal to
$$iW = i[X,Y] - i[JX,JY] + [X,JY] + [JX,Y]$$

Equating real and imaginary parts: $J[X,Y] - J[JX,JY] = [X,JY] + [JX,Y]$ real
$J[X,JY] + J[JX,Y] = -[X,Y] + [JX,JY]$ imaginary
imaginary $= J$ real so if real is true, then so is imaginary

Therefore when $J[X,Y] - J[JX,JY] - [X,JY] - [JX,Y] = 0$ for all $X,Y \in T^{(1,0)}(M)$ then the mfld is complex.

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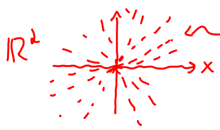
Examples of Complex Manifolds

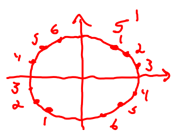
Obviously $\mathbb{C}^n$ is a complex mfld. It can be covered by one chart so it needs no transition functions.

For a less trivial example, first consider real projective space $\mathbb{RP}^n$. This can be realized as the set of lines passing through the origin in $\mathbb{R}^{n+1}$.



each line is a "point" in $\mathbb{RP}$, hence $\mathbb{RP}$ is 1D.

Another useful definition is $\mathbb{RP}^n = S^n$ w/ antipodal points identified



$x=y=z=\dots=0$

And finally, and most useful for us, we can start w/ $\mathbb{R}^{n+1} \setminus \{0\}$ and quotient by $(x,y,z,\dots) = \lambda(x,y,z,\dots)$ where λ is any nonzero real number, e.g. $x = 0.1x = -0.1x = 0.2x = -0.2x =$ etc.
This is really just a formal version of the first definition above.
Note we have to remove $\{0\}$ in order to get a well-defined quotient. (x,y,z) are "homogeneous" coord.

Let's show this is a manifold. Consider $\mathbb{RP}^2$ defined in $\mathbb{R}^3 \setminus \{0\}$. Consider open sets
$U_x = \{x \neq 0, y, z\}$, $U_y = \{x, y \neq 0, z\}$, $U_z = \{x, y, z \neq 0\}$
We have that $\mathbb{RP}^2 = U_x \cup U_y \cup U_z$
On each we can define inhomogeneous coordinates: $U_x = \underbrace{(1, \tfrac{y}{x}, \tfrac{z}{x})}_{n_x^i}$, $U_y = \underbrace{(\tfrac{x}{y}, 1, \tfrac{z}{y})}_{n_y^i}$, $U_z = \underbrace{(\tfrac{x}{z}, \tfrac{y}{z}, 1)}_{n_z^i}$
On an intersection, e.g. $U_x \cap U_y$ we know that $x \neq 0$ and $y \neq 0$ so we can convert from one coordinate system to the other w/ $\psi^i_{xy} : n^i_y(U_x \cap U_y) \to n^i_x(U_x \cap U_y)$
where ψ^i_{xy} is simply multiplication w/ $\frac{y}{x}$, i.e. $\frac{y}{x}(\frac{x}{y}, 1, \frac{z}{y}) = (1, \frac{y}{x}, \frac{z}{x})$
But this is clearly differentiable, so $\mathbb{RP}^2$ is a mfld.

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Now lets generalize this definition to $\mathbb{CP}^n$ by starting w/ $\mathbb{C}^{n+1} \setminus \{0\}$ and quotient w/ $(z_1, z_2, \cdots, z_{n+1}) = \lambda(z_1, z_2, \cdots, z_{n+1})$ where λ is any nonzero complex #.

Let's prove that this is a complex mfld. Proceeding as we did in the real case, consider open charts $U_\alpha = \{z_\alpha \neq 0\}$ where $\mathbb{CP}^n = \bigcup_{\alpha=1}^{n+1} U_\alpha$

Define inhomogeneous coordinates on each ; $x^\alpha_\mu = \frac{z_\mu}{z_\alpha}$

Then on an intersection we have $x^\alpha_\mu = \frac{z_\mu}{z_\alpha} = \frac{z_\mu}{z_\beta}\frac{z_\beta}{z_\alpha} = x^\beta_\mu \frac{z_\beta}{z_\alpha}$ so the transition functions are $\frac{z^\beta}{z^\alpha}$ which are clearly holomorphic. Hence $\mathbb{CP}^n$ is a complex mfld, and it so happens a compact one.

Other examples of complex mflds are submanifolds of complex manifolds. In particular, starting w/ the compact examples of $\mathbb{CP}^n$ we can generate submanifolds as the zero locus of a finite number of polynomial equations.

In $\mathbb{CP}^4$ we can use $z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0$ to generate the Fermat quintic.

As it will turn out, this is one example of a Calabi Yau three-fold.