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//-transport

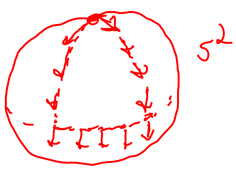
This extremely useful concept finds extensive application in GR and in defining complex mflds.

The idea begins w/ the observation that vectors (and dual vectors and all tensors for that matter) do not take values in the space itself, but rather in tangent spaces at each poin (or cotangent spaces or direct products of these).

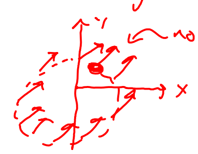
But for curved spaces especially, the tangent spaces may not align from point to point, so we have to be careful in "transporting" a vector around.

The idea of //-transport is the best approximation to keeping the vector // to itself as we move it around. While the changes are infinitesimal for infinitesimal displacements, they can add up.

note that the vector rotated by the time we got it back



whereas in $\mathbb{R}^2$



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Holonomy

As you learned / will learn in GR, one way to detect curvature in a space(time) is to check for nontrivial holonomies.

Consider a point on M and all closed paths through M. //-transport a vector around all of these, and the resulting set of transformations forms $\underbrace{Hol_p(M)}$.

actually independent of point p chosen so just $Hol(M)$

For connected Riemannian manifold (orientable) w/ metric connection, $Hol(M) \subseteq SO(n)$. One way to think about this is that a vector lives in $T(M)$ which at each point is a copy of $\mathbb{R}^n$. If we //-transport the vector around a closed loop we can imagine this as a transformation from $\mathbb{R}^n \to \mathbb{R}^n$ which preserves the length of the vector, i.e. rotations in $\mathbb{R}^n$ or $SO(n)$.

For special cases it can be simpler, e.g. for $\mathbb{R}^n$ $\nu_{\rho\alpha} = 0$, $Hol(M) = I$, i.e. in a flat space, all vectors are //-transported back to their original orientation.

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Kähler Manifolds

Kähler manifolds are essentially those for which $\parallel$-transport of a holomorphic vector, i.e. $v \in T^{(1,0)}(M)$, remains holomorphic.

Now recall that starting w/ a real manifold which locally looks like $\mathbb{R}^{2n}$, we complexified it to locally $\mathbb{C}^n$. The real tangent space was $\mathbb{R}^{2n}$ itself which we complexified to $\mathbb{C}^{2n}$, but then split into holomorphic and antiholomorphic subspaces each of complex dimension n.

So for a Kähler manifold of real dimension $2n$, $\parallel$-transport maps $\mathbb{C}^n \to \mathbb{C}^n$ and preserves the length of a vector. These transformations form $U(n)$, i.e. $Hol(M) \subseteq U(n)$.

					$n=3$	$n=4$ etc.
Let's compare:	$\mathbb{R}^{2n}$	$\Rightarrow SO(2n)$	$\Rightarrow \frac{1}{2} 2n(2n-1) = 2n^2 - n$	generators	15	28
	$\mathbb{C}^n$	$\Rightarrow U(n)$	$\Rightarrow n^2$	generators	9	16

So we see that being Kähler is a restriction (or rigidity) on the underlying real geometry.

It will turn out that for Calabi-Yaus, the restriction is even stronger, i.e. $Hol(M) \subseteq SU(n)$

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We are now in a position to state the conditions on a Calabi-Yau.

In 1954 Eugenio Calabi proposed that any compact Kähler manifold with vanishing first Chern class, i.e. $c_1(M) = \frac{1}{2\pi}\mathrm{Tr}(R)$, admits a unique metric for which $R_{\mu\nu} = 0$.

for holomorphic tangent bundle

Calabi proved the uniqueness of this metric, but it wasn't until 1977 that Shin Tung Yau proved the existence. Yau won the Fields medal in 1982 in part for this proof.

Note, while it is trivial that $R_{\mu\nu} = 0 \Rightarrow R^{\mu}{}_{\nu} = 0 \Rightarrow c_1(M) = \frac{1}{2\pi}\mathrm{Tr}(R) = 0$ it is by no means trivial to prove that $c_1(M) = \frac{1}{2\pi}\mathrm{Tr}(R) = 0 \Rightarrow R_{\mu\nu} = 0$.

What does $R_{\mu\nu} = 0$ mean? This is the condition of Ricci-flatness (as opposed to flat-flatness, i.e. $R^{\mu}{}_{\nu\lambda\rho} = 0$ ($\Rightarrow R_{\mu\nu} = 0$)).

Examples: $\mathbb{M}^n \Rightarrow R^{\mu}{}_{\nu\lambda\rho} = 0$, $AdS^5_R \times S^5_R \Rightarrow R^{\mu}{}_{\nu\lambda\rho} \neq 0$ but $R_{\mu\nu} = 0$

The importance is that any space w/ $R_{\mu\nu} = 0$ automatically satisfies Einstein's eqns. in vacuum, i.e. $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} = 0$.

We also have that $V^{\mu\prime} = V^{\mu} + \delta a^{\lambda\rho} R_{\lambda\rho}{}^{\mu}{}_{\nu} V^{\nu}$ which measures holonomy.

But for Kähler mflds, the $U(n) \simeq SU(n) \times U(1)$ near the identity w/ the $U(1)$ piece generated by $R_{\mu\nu}$. Hence for $R_{\mu\nu} = 0 \Rightarrow \mathrm{Hol} = SU(n)$.

This in turn leads to: For a Kähler mfld, $SU(n)$ holonomy $\Rightarrow$ Ricci-flat.