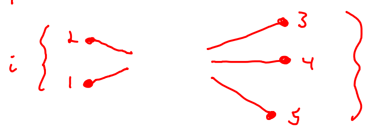
Sunday, March 10, 2019
6:10 PM

Consistency

Recall that in QFT we are often interested in computing transition amplitudes between an initial configuration of particles (localized excitations) and a final configuration:



f_k

different possible outcomes

Each of these particle states is built from the vacuum state $|0\rangle$ with field creation operators, e.g. $\phi(x_1)\phi(x_2)|0\rangle$

Then $M = \langle f | i \rangle = \langle 0 | \phi(x_5)\phi(x_4)\phi(x_3)\phi(x_2)\phi(x_1)|0\rangle = \underbrace{G_5(x_1, x_2, x_3, x_4, x_5)}_{\text{5-point correlation function}}$

$G_2(x_1, x_2)$ – Green's function
$G_2(x_1, x_2)$ for free theory is Feynman propagator

In terms of a path integral: $G_n(x_1, x_2, \ldots, x_n) = \dfrac{\int D\phi\, \phi(x_1)\phi(x_2)\cdots\phi(x_n)\, e^{\frac{i}{\hbar}S[\phi]}}{\int D\phi\, e^{\frac{i}{\hbar}S[\phi]}}$ } must be re-evaluated for each possible outcome

Alternatively, if we construct "the holy grail" of a QFT: $Z[J] = \int D\phi\, e^{\frac{i}{\hbar}S[\phi] + \frac{i}{\hbar}\int d^4x\, J(x)\phi(x)}$ then:

source current for $\phi(x)$

$$G_n(x_1, x_2, \ldots, x_n) = (-i\hbar)^n \frac{1}{Z[0]} \frac{\partial^n Z[J]}{\partial J(x_1)\,\partial J(x_2)\cdots\partial J(x_n)}\Bigg|_{J=0}$$

$Z[J]$ is a "generating function" because differentiating it generates correlation functions.
$Z[J]$ or sometimes just $Z[0]$ is also often called the "partition function" of the the theory both for the analogous role that $Z[J]$ plays, and for the fact that a Euclideanized $Z[0]$ is actually the partition function of a thermal system!

It turns out that the starting point for defining perturbative string theory is <u>not</u> a field theory in spacetime, but rather the relativistic quantum theory of particles in spacetime (even though we do use a 2D field theory to describe things from the world-sheet (WS) point of view)

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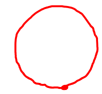
Consider the partition function for a particle of mass m_i in d-dimensions:



$$Z_{S^1}(m_i^2) = V_d \int \frac{d^dk}{(2\pi)^d} \int_0^\infty \frac{dl}{2l} e^{-(k^2+m_i^2)/2} \qquad l \text{ is the length of the circle}$$

$$= i V_d \int_0^\infty \frac{dl}{2l} (2\pi l)^{-\frac{d}{2}} e^{-m_i^2 l/2}$$

This quantity diverges for $l \to 0$, which reflects the typical high-energy (UV) divergences in QFT.

Let's see if making the point • into a string ○ helps. We will encode the new circle by θ. Additionally, we must include in our expression a sum over all vibrational modes of the string, i.e. string states, since they all come from the string itself (as opposed to different fields in QFT). So we are really interested in $\sum_i Z_{S^1}(m_i^2)$ in the case of a string, i.e. <u>all string states contribute</u>!

Combining l and θ into a complex combination $\tau = \frac{\theta}{2\pi} + \frac{il}{2\pi\alpha'}$ where α' = string length scale
$= \tau_1 + i\tau_2$

$$\sum_i Z_{S^1}(m_i^2) = i V_d \int_0^\infty \frac{dl}{2l} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} (2\pi l)^{-\frac{d}{2}} \sum_i \exp\left(-(h_i + \tilde{h}_i - 2)\frac{l}{\alpha'} + i(h_i - \tilde{h}_i)\theta\right)$$

h_i - right moving WS hamiltonian
$\tilde{h}_i$ - left moving WS hamiltonian

$$= i V_d \int_R \frac{d\tau d\bar{\tau}}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-\frac{d}{2}} \sum_i q^{h_i - 1} \bar{q}^{\tilde{h}_i - 1} \qquad q = e^{2\pi i \tau}$$

The integration region R includes $|\tau_1| = \left|\frac{\theta}{2\pi}\right| < \frac{1}{2}$, $\tau_2 = \frac{l}{2\pi\alpha'} > 0$

But this has made the divergence even worse! We still have a divergence as $l \to 0$, but now it is summed over all string states w/ contributions of the same sign (so no cancellations).

Did string theory make it worse?!

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We weren't doing string theory. What would the answer in string theory be?

$$Z_{T^2} =: V_d \int_F \frac{d\tau d\bar{\tau}}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-\frac{d}{2}} \sum_i q^{h_i - 1} \bar{q}^{\bar{h}_i - 1}$$

The only difference is the integration region F vs. R. What is F?

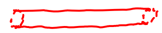
Recall that the WS theory of the string (the Polyakov action) enjoyed a diff $\times$ Weyl symmetry

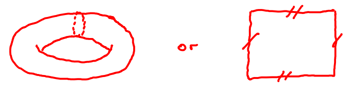
diff : $(\sigma^1, \sigma^2) \to (\sigma^{1\prime}, \sigma^{2\prime})$ WS coordinate redefinitions

Weyl: $g_{ab} \to e^{2\omega(\sigma^a)} g_{ab}$ local scale invariance of WS metric

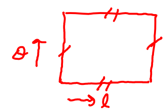
} We used these 3 to fix $g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in order to quantize. Confirming they were non-anomolous led to critical dimension.

Note: Both are continuous sets of transformations, i.e. they can be connected to $\mathbb{I}$.

That discussion played out for free strings, i.e. WS in the shape of an ∞ cylinder 

But now we are considering a different WS topology, i.e. T^2 

In this case, the diff $\times$ Weyl symmetry includes a discrete, disconnected set of transformations, i.e "large coordinate transf"

What are they? Recall $\tau = \frac{\theta}{2\pi} + \frac{i\ell}{2\pi\alpha'}$ where θ and ℓ describe the two "cycles" of the torus 

It turns out that $\underbrace{\tau \to \tau' = \frac{a\tau + b}{c\tau + d}}$ for integer a, b, c, d w/ $ad - bc = 1$ (the modular group) is also a symmetry.

"generated" by $\underbrace{\tau \to \tau' = \tau + 1}_{\substack{a = b = d = 1 \\ c = 0}}$ and $\underbrace{\tau \to \tau' = -\frac{1}{\tau}}_{\substack{a = d = 0 \\ b = -c = \pm 1}}$

From invariance under $\tau \to -\frac{1}{\tau}$, it is clear that any $|\tau| < 1$ can be replaced w/ a $|\tau'| > 1$, i.e. $\tau' = -\frac{1}{\tau}$.

But this means that our integration region F should only include $|\tau| > 1$ (and $|\tau_1| < \frac{1}{2}$).

But $|\tau| = \frac{\theta^2}{4\pi^2} + \frac{\ell^2}{4\pi^2\alpha'^2} > 1 \Rightarrow \ell^2 > 4\pi^2\alpha'^2 - \alpha'^2 \underbrace{\theta^2}_{< \frac{1}{4}} \Rightarrow \ell^2_{min} = 4\pi^2\alpha'^2 - \frac{1}{4}\alpha'^2 > 0$!!

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Two take-away messages here:

1. The finiteness of string theory (for closed oriented strings at least) depends critically on the modular invariance of the partition function (which can be suitably generalized by adding vertex operators).
2. The partition function includes, and in fact relies upon the contribution of all string states compatible with the WS topology.

Both of these make it seem like finding consistent string theories could be very tricky, but actually there is a recipe called "orbifolding" which does the job, and leads to a lot of insights. That will be the focus of our next talk.