Review I:

For closed oriented string, finiteness at 1-loop requires:

\[ z \rightarrow z + \frac{i\pi}{2} \Rightarrow z \rightarrow z + 1, \quad z \rightarrow z - \frac{i\pi}{2} \]

b) contributions from all closed string states, i.e. all solutions to the dR field theory and prescribed boundary conditions.

Review II:

A Lagrangian or action may exhibit a symmetry, such as a solution to the e.o.m. need not show that symmetry.

\[ U(1,1) \quad \text{is symmetric under relations} \quad \phi_1, \phi_2 \quad \text{in } \phi_1, \phi_2 \text{ plane} \]

\[ \text{This solution } \phi = (\phi_1, \phi_2) = (0, 0), \quad \text{is not invariant} \]

\[ \phi \]

Review III:

Requiring spinors, fermions and no topology \( \Rightarrow \) 5 supersymmetric theories in \( \mathbb{M}^1 \):

\[ N = 32 \quad \text{IIA, IIB} \quad \text{or} \quad \text{III} \]

\[ N = 16 \quad \text{IIB (E8 x E8), Heterotic (SO(32))} \]

\[ \text{Type I} \quad \text{(No open strings)} \]

We are going to look at some examples between these from a powerful tool called "twisting" which gives a new theory on \( \mathbb{M}^1 \).

We will then look at other examples of twisting which can change the underlying geometry and/or require new d.o.f.
Start w/ any consistent string theory based on oriented closed strings as a $N=1$ partition function.

Consulting the worldsheet action we may find some discrete symmetry transformation $g$ which leaves $\mathcal{D}$ invariant.

We may consider projecting the spectrum of the theory by $\mathcal{D} = \hat{\mathcal{D}}_\tau (1 + \hat{\mathcal{D}}_\tau) \mathcal{D}$ where $\hat{\mathcal{D}}_\tau (1 + \hat{\mathcal{D}}_\tau)$ is an integer, i.e. $\hat{\mathcal{D}}_\tau (1 + \hat{\mathcal{D}}_\tau) \in \mathbb{Z}$.

Projecting the partition function we find:

$$Z_{\mathcal{D}} \rightarrow Z_{\mathcal{D}^g} \left[ \mathcal{D} \right] = Z_{\mathcal{D}} \left[ \mathcal{D} + Z_{\mathcal{D}} (1 + \hat{\mathcal{D}}_\tau) \right] = \frac{1}{Z_{\mathcal{D}}} + \frac{1}{Z_{\mathcal{D}} (1 + \hat{\mathcal{D}}_\tau)}$$

This means when we join the two ends of the cylinder to make $\mathcal{D}$ we "twist" or operate on using $g$.

We know that $\frac{1}{Z_{\mathcal{D}} (1 + \hat{\mathcal{D}}_\tau)}$ is $N=1$, but in general we do not expect $Z_{\mathcal{D}^g} (1 + \hat{\mathcal{D}}_\tau)$ to be. One reason for this is that by projecting $\mathcal{D}$, we are throwing away some string states, but we know we need them all for $N=1$.

Thus if $Z_{\mathcal{D}^g} (1 + \hat{\mathcal{D}}_\tau)$ is $N=1$, then $Z_{\mathcal{D}^g} (1 + \hat{\mathcal{D}}_\tau)$ can be.

Thus on two approaches to fixing this:

a) Take $Z_{\mathcal{D}^g} (1 + \hat{\mathcal{D}}_\tau)$ and keep applying $\mathcal{D} + \frac{1}{Z_{\mathcal{D}^g} (1 + \hat{\mathcal{D}}_\tau)}$ until we generate new terms which together with $Z_{\mathcal{D}^g} (1 + \hat{\mathcal{D}}_\tau)$ form a closed (closed invariant) net.

b) Think about new string states that should be added, do so and then project them as well.

Both approaches get often get to the same result, and indeed route (b) was used to "explain" the new terms required by (a).
Examples:

IIA twisted by an action on W5 fermions gives $\mathbb{Z}_3$

Het (E8xE8) twisted by an action on gauge fermions gives Het (E8xE8)

Perhaps more interestingly consider the symmetry group of an action based on $X^{X^{X^{X^{X^{X^{X^{X^A}}}}}}}$, i.e. Poincaré in 0-dim.